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I. On a New Geometrical Theory of the Diffraction-Figures observed in the Heliometer.

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(Plate I).

CONTENTS.

SECTION I.—INTRODUCTION.

A mathematical theory of the form of the diffraction figures of the Fraunhofer class due to a semi-circular aperture was first developed by Bruns* who gave the formulae for the intensity of the light at any point in the neighbourhood of the focus of a heliometer objective. These formulae, however, demanded a very large amount of arithmetical calculation before they could be of any practical use in the determination of the form of the diffraction pattern, and the further work involved in this appears to

* Bruns, "Über die Beugungsfiguren des Heliometer-Objectives," 'Astr. Nachr.,' Bd. CIV, No. 2743.

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have remained unattempted till 1909 when Everitt * succeeded in carrying out the numerical computations and finding the form of the contour-lines of equal illumination in the neighbourhood of the focus in the diffraction pattern. The process adopted by him was extremely elaborate, and involved the use of a Brunsviga calculator for part of the numerical work, of a Coradi co-ordinatograph, and of an integrator and planimeter for semi-graphical and mechanical integration. While Everitt's paper is of undoubted value in view of the painstaking thoroughness and accuracy of his work, the method adopted by him is unsatisfactory from a physical point of view, as it is essentially numerical in spirit, and leaves the reader without a clear conception of how precisely the peculiar configuration of the pattern arises from the semi-circular form of the aperture. Everitt's work was also of somewhat limited scope, as the calculation of intensities was confined to a relatively small region surrounding the position of the geometrical focus.

It will be shown in the present paper how the theoretical determination of the form of the diffraction figures in the heliometer can be carried out without appreciable loss of accuracy by a method which is geometrical in character and is far less laborious than that adopted by Everitt. Not merely is the method simpler, but it has also the advantage of enabling the form of the pattern to be readily determined for a region of any desired area surrounding the focus. The geometrical form of the pattern deduced by this method is in close agreement with experiment, and shows certain features at large angular deviations from the focus which were unsuspected by Everitt and which indeed make necessary a modification of the description of the diffraction-figures published by him.

To illustrate the novel features observed by the author, photographs of the diffraction pattern with relatively long exposures have been secured, one of which enlarged to a suitable size is reproduced in Fig. 1 (Plate). For purpose of comparison, a photograph taken with a comparatively small exposure is reproduced in

* Everitt, "Diffraction Figures due to the Heliometer," 'Proc. of the Royal Society of Lond.' Series A, Vol. 83; 1910.

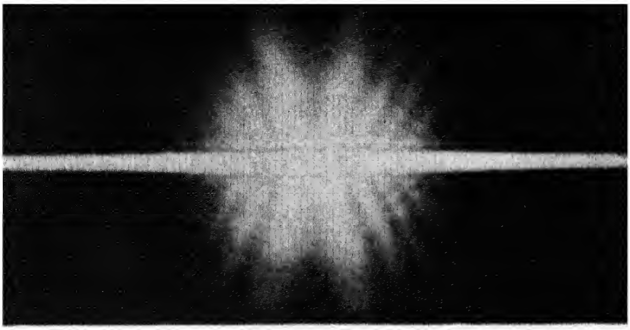

Fig. 1

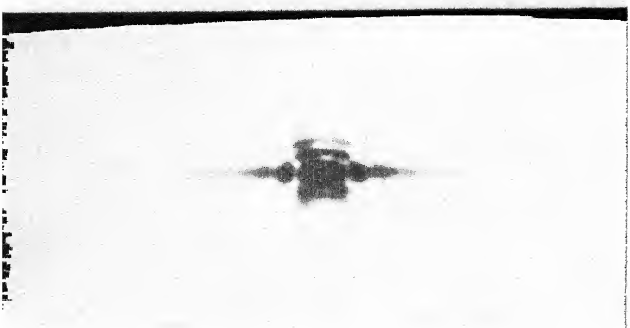

Fig. 2

Diffraction Figures observed in the Heliometer.

Fig. 2, the picture in this case being very similar to those published in Everitt's paper.* It will be noticed that Fig. 1 (Plate) shows the prolongation of the transverse streamers as *concave* outwards, a feature which fails to appear in the relatively limited region covered by the picture in Fig. 2 (Plate). The work has also brought to light another important feature not mentioned by Everitt, that the fluctuations of intensity along the long horizontal ray that appears crossing the pattern decrease both relatively and absolutely at large angular deviations.

Section II.—Principle of the Geometrical Theory.

The ordinary treatment of diffraction phenomena, both of the Fresnel and of the Fraunhofer class, proceeds by expressing the effect at any point in the field in terms of a surface integral taken over the area of the aperture, in other words, as the resultant of a collection of secondary sources of light situated over the whole area of the boundary. A considerable simplification is, however, effected by transforming the surface integral into a line integral, in other words, by regarding the effect at any point in the field as due to a linear source of light situated along the boundary of the aperture, and the linear source in turn can be replaced by a finite number of point sources of light, generally two, sometimes three or more, having appropriate phases and situated at certain points on the boundary. The position and intensity of these point-sources on the boundary is generally not fixed, but varies with the direction of the diffracted light. In other words, corresponding to each point in the focal plane at which the diffraction pattern is formed, we have certain points on the boundary which principally contribute to the luminous effect at the point of observation, and the whole of the diffraction pattern may be simply regarded as an interference pattern due to a finite number of light-sources of *variable position* situated on the boundary of the aperture.

The foregoing method of regarding diffraction is of con-

* And also in a recent paper by Gordon, 'Proc. of the Phys. Soc. of Lond., October 1912.

siderable utility in forming a mental picture of the way in which the phenomena due to any aperture of arbitary form arise, and explaining the geometrical relationship between the form of the aperture and the form of the diffraction pattern produced by it. This has been emphasised in a recent paper by me on large-angle diffraction by curvilinear boundaries.* So far from being merely a convenient mathematical fiction, the existence of sources of light situated at specific points on a curvilinear diffracting boundary may be directly verified by observation or photography, as has been shown in a recent paper.† For this purpose, the aperture may be viewed by the aid of the diffracted light only, admitted into an observing telescope through a small hole in a screen otherwise completely cutting off the light reaching the focal plane [as in the well-known method of the Foucalt test]. When the luminous emission from the boundary of a semi-circular aperture is observed in the foregoing manner, the following phenomena are noticed : in general, *three* points on the boundary are seen to be luminous: two of equal but small intensity are situated on the two corners, and a third and more intense one is situated on the arc of the semi-circle at such point that the corresponding radius of the semi-circle is parallel to the line joining the centre of the focal plane with the orifice through which the diffracted light enters the observing telescope. There are, however, two exceptions to this general rule : when the orifice is situated at any point on the long horizontal ray in the diffraction-pattern running perpendicular to the diameter of the semi-circle, the whole of the diameter as well as the mid-point of the arc appear luminous. On the other hand, if the orifice in the focal plane is situated on a line drawn parallel to the diameter of the semi-circle, we have only two light sources visible which are situated respectively at the two extremities of the diameter and appear brilliantly luminous. The transition from the phenomena as observed in the two special portions of the orifice in the focal plane to that seen in the general case when it is placed in any arbitary position takes place in a fairly sudden manner.

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* S. K. Mitra, Phil. Mag., Sept. 1919.

† Dr. S. K. Banerjee, Phil. Mag., Jan. 1919.

Section III.—Intensity in the Diffraction-Pattern in the two Principal Directions.

The illumination at any point in the focal plane lying in directions parallel and perpendicular to the diameter of the semi-circular aperture may be very readily calculated as follows:

Case I: Parallel to the diameter.

In this case we may begin by dividing the semi-circle into narrow strips parallel to the diameter and adding up the effects of these strips.

To obtain the effect of one of these strips we observe that the expression for disturbance due to an elementary portion $dx\,dy$ of the strip (of width dy) may be written as

$$A' \sin\left(wt - \frac{2\pi}{\lambda}\frac{r}{f}x\right) dx\,dy,$$

(the X axis being taken parallel to the strips), where r is the distance of the point of observation from the centre of the pattern in the focal plane, f the focal length of the lens, and A' is a const. such that if I be the intensity at the centre of the pattern, $\sqrt{I} = A' \times$ *area of the aperture.* The above when integrated between the limits x_1 and x_2, the co-ordinates of the two extremities of the strip gives us

$$\frac{\lambda f A' dy}{2\pi r}\left[\cos\left(wt - \frac{2\pi}{\lambda}\frac{r}{f}x_2\right) - \cos\left(wt - \frac{2\pi}{\lambda}\frac{r}{f}x_1\right)\right]$$

$$= \frac{2dy}{\pi R\delta}\left[\sin\left(wt - \frac{2}{\lambda}\frac{r\pi}{f}x_1 - \frac{\pi}{2}\right) + \sin\left(wt - \frac{2\pi}{\lambda}\frac{r}{f}x_2 + \frac{\pi}{2}\right)\right]$$

where $$\delta = \frac{2\pi}{\lambda}\frac{r}{f}R \text{ and } A' = \frac{2}{\pi R^2},$$

the intensity at the centre of the pattern being taken to be unity. R is the radius of the semicircular arc.

This shows that the effect of each strip may be replaced by that of two light-sources of equal strength and opposite phase situated at its extremities. (It will be noticed that the strength of the sources is independent of the length of the strip.) The effect of the whole aperture accordingly reduces to a linear distribution of light sources along the semicircular arc. If the aper-

ture had been a complete circle instead of being a semicircle, there would have been a precisely similar and symmetrical distribution along the other half of the circle. It follows from this that the resultant amplitude at any point due to the distribution of sources on the boundary of the semicircular aperture is half that for the case of the circular aperture, the intensity being one-fourth. *The positions of the maxima and minima of illumination along this direction are accordingly identical with those obtained in the well-known investigation for the case of a circular aperture.*

Case II : Perpendicular to the diameter.

In this case, the semicircular aperture may be imagined divided up into strips perpendicular to the diameter. Proceeding in the same manner as before, the effect of the whole aperture reduces to a linear distribution of sources along the straight and the curved parts of the boundary, which it is convenient to consider separately.

For the straight portion of the boundary we have a distribution of sources of amplitude

$$\frac{2dy}{\pi R\delta} \sin\left(wt - \frac{2\pi}{\lambda}\frac{r}{f}x_1 - \frac{\pi}{2}\right).$$

The X axis being taken parallel to the strips (i.e. perpendicular to the diameter), x_1 is constant, and the phases of the contributions of the different parts of the boundary are all the same. The effect due to the whole diameter is thus obviously

$$\frac{2 \times 2R}{\pi R\delta} \sin\left(wt - \frac{2\pi}{\lambda}\frac{r}{f}x_1 - \frac{\pi}{2}\right),$$

or taking the Y axis along the diameter (i.e. $x_1 = o$), it is equal to

$$\frac{4}{\pi\delta} \sin\left(wt - \frac{\pi}{2}\right).$$

For the curved portion of the boundary we have a similar distribution of sources of amplitude

$$\frac{2dy}{\pi R\delta} \sin\left(wt - \frac{2\pi}{\lambda}\frac{r}{f}x_2 + \frac{\pi}{2}\right).$$

In this case, however, x_2 being different for different parts of the boundary, the phases of the contributions from different parts

would vary rapidly and would be stationary only at the mid-point of the arc. The resultant effect of the sources situated along the semicircular arc may be deduced at once from the consideration that if a precisely similar distribution of sources on a semicircular arc be placed symmetrically on the other side of the diameter, the joint effect of the two together should give us the well-known expression for the case of a complete circular aperture which is

$$4\frac{J_1(\delta)}{\delta}\sin wt\,^*$$

Using semi-convergent expansions we have

$$J_1(\delta)=\sqrt{\frac{2}{\pi\delta}}\left[\sin\left(\delta-\frac{\pi}{4}\right)+\frac{3}{8}\frac{\cos\left(\delta-\frac{\pi}{4}\right)}{\delta}\right]$$

$$=\sqrt{\frac{2}{\pi\delta}}\sin\left(\delta-\frac{\pi}{4}+\frac{3}{8\delta}\right)$$

approximately, and the expression for the disturbance due to the complete circular boundary is accordingly

$$\frac{4}{\delta}\sqrt{\frac{2}{\pi\delta}}\sin\left(\delta-\frac{\pi}{4}+\frac{3}{8\delta}\right)\sin wt$$

This disturbance may be resolved into two parts,

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\sin\left(wt-\delta+3\frac{\pi}{4}-\frac{3}{8\delta}\right)$$

and $$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\sin\left(wt+\delta-3\frac{\pi}{4}+\frac{3}{8\delta}\right)$$

each due to one-half of the circular boundary. The effect due to the distribution of sources situated along the semicircular arc in this case is thus equal to

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\sin\left(wt-\delta+3\frac{\pi}{4}-\frac{3}{8\delta}\right).$$

* This is usually written $2\frac{J_1(\delta)}{\delta}\sin wt$, taking the intensity at the centre of the pattern due to the *circular* aperture to be unity. Since we have taken the intensity at the centre of the pattern due to the *semicircular* aperture to be unity the expression is $4\frac{J_1(\delta)}{\delta}\sin wt$.

The complete expression for the disturbance due to the semi-circular aperture is the sum of that due to the curved and straight parts of the boundary, that is equal to

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\sin\left(wt+3\frac{\pi}{4}-\delta-\frac{3}{8\delta}\right)+\frac{4}{\pi\delta}\sin\left(wt-\frac{\pi}{2}\right).$$

It is obvious that of these two terms, the latter varying as δ^{-1} ultimately predominates for increasing values of δ. The intensity in the diffraction pattern in the direction perpendicular to the diameter thus ultimately varies as δ^{-2}, while in a direction parallel to the diameter, it decreases much more rapidly in the ratio of δ^{-3}. Thus, we have in the diffraction pattern a long bright ray running perpendicular to the diameter of the semi-circle. Further, in the direction of the bright ray, not merely do the absolute intensities diminish, but the ratio of the intensities at maxima and minima also tends to approach unity, in other

	Author's Values.		Everitt's Values.	
	δ	H	δ	H
Centre ..	0·0	1·000	0·0	1·000
1st Min. ..	7·06	0·00897	7·1	0·009
1st Max. ..	9·65	0·0324	9·7	0·033
2nd Min. ..	13·40	0·00392	13·4	0·00397
2nd Max. ..	16·09	0·01050	16·0	0·01057
3rd Min. ..	19·69	0·00216	19·45*	0·00257*
3rd Max. ..	22·47	0·00506	22·45	0·00503
4th Min. ..	25·98	0·00137	25·95	0·00137
4th Max. ..	28·80	0·00294	28·85	0·00296
5th Min. ..	32·26	0·00094	32·25	0·00095
5th Max. ..	35·15	0·00191	35·15	0·00193
6th Min. ..	38·54	0·00069	38·50	0·00068
6th Max. ..	41·42	0·001342		
7th Min. ..	44·83	0·000535		
7th Max. ..	47·73	0·000980		
8th Min. ..	51·11	0·000422		
8th Max. ..	54·03	0·000763		
9th Min. ..	57·39	0·000340		
9th Max. ..	60·32	0·000599		
10th Min. ..	63·67	0·000284		
10th Max. ..	66·61	0·000469		
11th Min. ..	69·95	0·000240		
11th Max. ..	72·92	0·000402		

* Everitt's results seem to be in error.

words, the fluctuations of intensity along the bright ray tend to diminish both relatively and absolutely. The formula given above enables the intensity to be readily calculated for any value of δ with great ease. In the following table, several values and positions of the maxima and minima along the bright ray are shown, and alongside for comparison Everitt's values for the first few maxima and minima are also given. It will be seen that the same degree of accuracy as that obtained by Everitt is obtainable by this method with far less labour.

Section IV.—General Configuration of the Pattern.

We now proceed to prove that (excluding the two principal directions) the whole effect of the aperture might be regarded as equivalent to three sources of appropriate phase and intensity two of which are situated at the two corners and the other on the

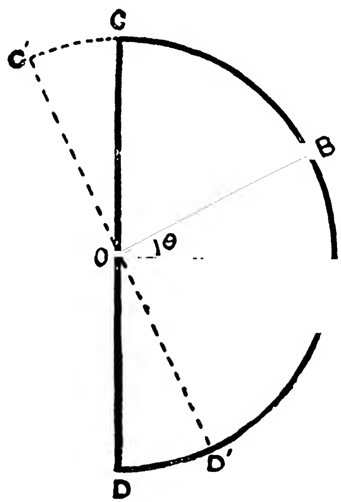

Fig. 3.

semicircular arc. The general configuration of the pattern will also be deduced on this basis.

Divide the semi-circular aperture into strips parallel to the line joining the point of observation to the centre of the pattern in the focal plane (as for instance parallel to OB in Fig. 3). As

before, the effect of all these strips can be reduced to a linear distribution of sources situated on the boundary of the aperture. The phases of the contributions from different parts of the straight portion of the boundary change regularly from C to D. We can accordingly replace the effect by a source situated at the centre O, or by two sources of opposite phase and equal amplitude situated at C and D. The phase of the resultant source at O being $-\pi/2$ (the elementary contributions being retarded by that amount on account of strip division), the phases of the component sources at C and D would be $-\delta \sin \theta$ and $\delta \sin \theta - \pi$ (the phases of the *contributions* from C and D are $-\delta \sin \theta$ and $\delta \sin \theta$ respectively, if the phase of contribution from O is taken to be zero, that from B being $-\delta$. $\theta = \pi/2 - \angle COB$). For the curved portion of the boundary, produce the semi-circle DBC to meet $C'D'$ drawn perpendicular to OB. We observe that the effect of the arc CBD is obviously equivalent to the effect of

(the semicircular arc $D'BC'$)—(the arc $C'C$)+(the arc DD').

The semicircular arc $C'B'D'$, we have already seen, can be replaced by a source of proper phase $(-\delta + \frac{3}{4}\pi)$ and amplitude situated at B. The effect of the arcs CC' and DD' are replaceable by two sources at C and D respectively of equal amplitude (since the arcs are of equal length), whose phases are the same as the phases of the components at these points due to the straight portion of the boundary CD.* We thus have for the whole aper-

* This is evident from the following considerations: The amplitude of an elementary portion ds of the boundary being proportional to the width of the corresponding strip, is proportional to $ds \cos \psi$ (ψ=angle between the normal to ds and the length of the strip). The amplitude of an elementary contribution consequently changes from a maximum at B to zero value at C, while the phase advances from $-\delta$ to $-\delta \sin \theta$. If such a system of vibrations of continually decreasing amplitude and advancing phase is compounded, the phase of the resultant is approximately the same as that of the first vibration plus $\frac{\pi}{2}$. (This result is always true except when the phase of the first vibration passes through a maximum or minimum value. The above arguments are easily understood if the vibrations are compounded graphically. *Cf.* Preston, Theory of Light (third edition), page 250, fig. 120. The resultant effect due to the two arcs are therefore two sources at C and D of phases $-\delta \sin \theta + \pi/2$ and $\delta \sin \theta - \pi/2$ (unless C or D lie close to B, where the phase is a minimum). Remembering that the phases of the elementary portions of the arcs CC' and DD' are advanced and retarded respectively by $\pi/2$ (on

ture a source at B (due to the semicircular arc $D'BC'$), and a pair of sources at C and D (due both to the straight portion CD and the arcs CC' and DD'), their phases being

$$-\delta+\tfrac{3}{4}\pi \quad \text{at } B$$
$$-\delta\sin\theta \quad \text{at } C$$
$$\delta\sin\theta-\pi \quad \text{at } D.$$

(The source B is of course of variable position: its point of situation on DBC depending on the point of observation in the focal plane as explained before).

In order to find the nature of the diffraction pattern let us take any two of the sources and find out the positions of constant phase difference in the focal plane. Thus, for the sources C and B there would be maximum of illumination when

$$-\delta\sin\theta+\delta-\tfrac{3}{4}\pi=2\pi,\ 4\pi,\ \text{etc., i.e. } \delta-\delta\sin\theta=\text{const.}$$

This would give us in the focal plane a series of parabolas branching upwards, of common axis lying parallel to the diameter of the semicircle. Similarly the sources D and B would give us another set of parabolas, their axis the same as the former set, but the parabolas themselves branching downward.

$$\delta+\delta\sin\theta=\text{const.}$$

Also for the sources C and D, the condition of the maximum illumination is

$$\delta\sin\theta+\delta\sin\theta-\pi=2\pi,\ 4\pi,\ \text{etc., i.e. } \delta\sin\theta=\text{const.}$$

This gives us a number of straight lines in the focal plane running perpendicular to the diameter of the semi-circle, i.e. parallel to the horizontal bright ray. These features, viz. two sets of parabolas branching upwards and downwards, and a set of straight lines (a few of them) running perpendicular to the axis of the parabolas are shown in the drawings, Figs. 4 and 5. It will be seen that Fig. 4 closely reproduces the features appearing in the long-exposure photograph of the author in the Plate. (Fig. 1).

There is an alternative way of regarding the "interference"

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account of strip division), and that the effect of CC' is to be subtracted, the phases of the resultants at C and D finally become $-\delta\sin\theta$ and $\delta\sin\theta-\pi$.

pattern in the focal plane which is both useful and instructive. The phase of the two sources at C and D being

$$-\delta \sin \theta \text{ and } \delta \sin \theta - \pi,$$

these when combined together give a source of fluctuating amplitude * at O proportional to $\sin(\delta \sin \theta)$, and phase $-\frac{\pi}{2}$.

The "interference" pattern in the focal plane can be regarded as due to the superposition of a source of fluctuating amplitude at O, and a source of variable position at B on the semicircular arc (the line OB being parallel to the line joining the point of observation in the focal plane to the centre of the pattern). Since the

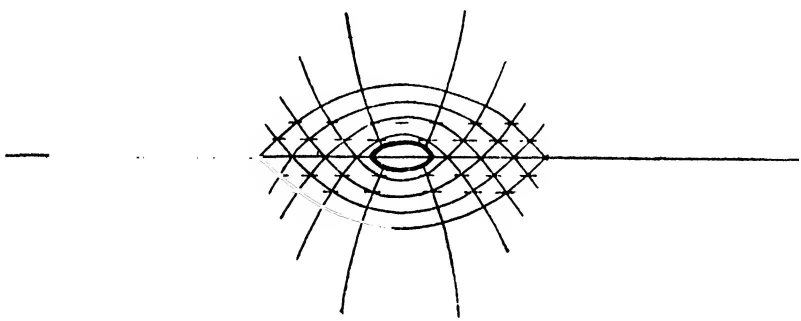


FIG. 4.

phases of these sources are independent of θ (i.e. direction of observation in the focal plane), the lines of equal phase will be circles round the centre of the pattern. The real maxima will however be at those positions in the focal plane where both O and B agree in phase, and at the same time the amplitude of the source O is a maximum (the amplitude of the other source being independent of θ).

* The part of the amplitude of the source at O due to the straight portion is as will be seen in the following section really proportional to $\frac{\sin(\delta \sin \theta)}{\delta \sin \theta}$, so that the positions of the maxima are slightly different from that given by $\delta \sin \theta = (2n+1)\frac{\pi}{2}$, when θ is small. This has been taken into account in drawing fig. 5. For small values of θ the other part of the amplitude (due to the arcs CC' and DD') is very small, so that the positions of maxima coincides with the maximum values of $\frac{\sin(\delta \sin \theta)}{\delta \sin \theta}$.

The first condition is given by

$$-\frac{\pi}{2}-\left(-\delta-\frac{3}{8\delta}+3\frac{\pi}{4}\right)=2\pi,\ 4\pi,\ \text{etc.}$$

and the second by

$$\delta\sin\theta=\frac{\pi}{2},\ 5\frac{\pi}{2},\ 9\frac{\pi}{2},\ \text{etc.}$$

The first gives us a series of circles, and the second a series of parallel straight lines. The loci of maximum illumination lie on the intersections of these two sets of curves and are obviously parabolas. Fig. 5 shows the forms of the curves as obtained from the above considerations. The features appearing in the drawing

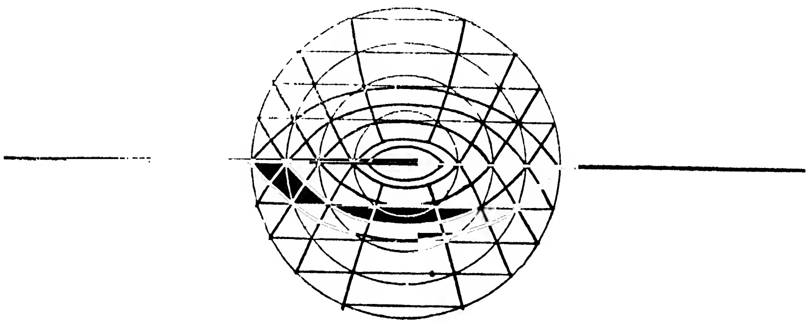


FIG. 5.

are also clearly noticeable in the long-exposure photograph in the Plate.

SECTION V.—THE DETERMINATION OF THE INTENSITY OF ILLUMINATION AT ANY POINT OF THE PATTERN.

The effect at any point in the focal plane being equal to the sum of two sources, one at O and another at a corresponding point B on the arc, we proceed to find the amplitude of these sources.

The amplitude of the source at B due to the semicircular arc $D'BC'$, we have already found to be

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}},$$

that of the source at O being composed of two parts, one due to the straight portion CD and the other due to the arcs CC' and DD', we can find their resultants separately.

The contribution from an elementary portion ds of CD being

$$\frac{2ds}{\pi R\delta}\cos\theta\sin\left(wt-\frac{2\pi}{\lambda}\frac{s\,.\sin\theta}{f}-\frac{\pi}{2}\right)$$

its total effect is found by integrating the above expression between limits R and $-R$ (*vide* Fig. 3. $OC=OD=R$ and $s=$ distance of ds from O) to be

$$\frac{4}{\pi\delta}\cos\theta\frac{\sin(\delta\sin\theta)}{\delta\sin\theta}\sin\left(wt-\frac{\pi}{2}\right).$$

The resultant effect due to the arc CC' (or DD') can be found by dividing it into half-period "zones" and adding up the effects of all of these. This can be done in the following way :—

Beginning from B (fig. 3) divide the arc BC' into small parts such that the phases of the resultant contributions from successive parts (with reference to the point of observation in the focal plane) differ by π, that of the first one being the same as the resultant of the whole arc BC'. This is easily done by following a method exactly analogous to that employed by Schuster * in finding the resultant effect of a plane wave, and the total effect of the arc BC', i.e. half the semicircular arc (except the first 'zone') can be expressed in the form

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\left[(\sqrt{7}-\sqrt{3})-(\sqrt{11}-\sqrt{7}+(\sqrt{13}-\sqrt{11})+\ldots\ldots\right|$$

Knowing the effect due to half the semicircle to be

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\times\cdot5,$$

the effect of the first 'zone', since it is of opposite sign to that of the resultant of the rest of the arc, is easily seen to be

$$\cdot6725\times\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}$$

* Schuster, Theory of Optics (2nd edition, 1909), page 95.

the sum of the above series within bracket being ·1725. The effect of half the semicircular arc can thus be expressed in the form

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\,(\cdot 6725-\cdot 2908+\cdot 2135-\cdot 1771 \text{ etc.}).$$

By omitting the first term or the first two terms or the first three terms from the above series, the effect of corresponding portion of the arc as given in the following table can be found out:—

LENGTH OF THE ARC.	AMPLITUDE.
Half-semicircle $\delta-\delta\sin\theta=0$	$\cdot 5\times\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}$
,, except 1st zone $\delta-\delta\sin\theta=3\frac{\pi}{4}$	·1725 × ,,
,, except first two zones $-\delta\sin\theta=\frac{7\pi}{4}$	·1183 × ,,
,, except first three ,, $\delta-\delta\sin\theta=\frac{11\pi}{4}$	·0925 × ,,
etc.	etc.

If a curve be now plotted between different values of $\delta-\delta\sin\theta$ (corresponding to different lengths of the arc) and the corresponding effect in amplitude as found from the above table, we can obtain from that the effect in amplitude for *any* value of $\delta-\delta\sin\theta$, i.e. for *any* length of the arc CC' (or DD'). Designating the amplitude thus found by

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\cdot f(\delta-\delta\sin\theta),$$

the expressions for the resultant sources at C or D due to the arcs CC' and DD' taking their proper phases into account as found in § 4 are at once obtained.

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\,f(\delta-\delta\sin\theta)\sin(wt-\delta\sin\theta)$$

and

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\, f(\delta-\delta\sin\theta)\sin(wt+\delta\sin\theta-\pi).$$

These when combined give us a resultant source

$$2f(\delta-\delta\sin\theta)\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\sin\left(\delta\sin\theta\right)\sin\left(wt-\frac{\pi}{2}\right)$$

at O.

We thus have the complete expression for amplitude for any point in the focal plane:

$$\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\sin\left(wt+3\frac{\pi}{4}-\delta-\frac{3}{8\delta}\right)$$

$$+\left[\frac{4}{\pi\delta}\cos\theta\frac{\sin(\delta\sin\theta)}{\delta\sin\theta}+2f(\delta-\delta\sin\theta)\frac{2}{\delta}\sqrt{\frac{2}{\pi\delta}}\sin(\delta\sin\theta)\right]\sin\left(wt-\frac{\pi}{2}\right).$$

In the following table are given the values of intensity H for various values of θ and δ, in the focal plane together with Everitt's values for comparison. The latter were obtained from his contour diagram of equal intensity.

The agreement can be seen to be satisfactory.

$\delta=7{\cdot}01$

	θ	0°	9°42′	33°	50°	63°	73°	90°
Author ..	H	·0092	·0050	·0163	·022	·007	·0033	·0000
Everitt ..	H	0098	·004	·015	·015	·006	·002	·0000

$\delta=10{\cdot}4$

	θ	0°	13°48′	32°	41°48′	55°12′	70°	90°
Author ..	H	·0291	·0068	·0007	·0040	·0045	·0015	·00012
Everitt ..	H	·0291	·006	·001	·004	·004	·001	·00011

$\delta = 13{\cdot}32$

	θ	0°	15°48′	28°6′	39°42′	50°48′	60°18′	69°30′	80°24′	90°
Author	H	·0039	·0021	·0010	·00038	·0020	·0022	·0012	·00036	0000
Everitt	H	·004	·002	·001	·0005	·002	·002	·001	·0002	·0000

$\delta = 16{\cdot}47$

	θ	0°	5°30′	21°48′	31°24′	48°48′	56°	63°42′	90°
Author	H	·0102	·0059	.00052	·0011	·00049	·0010	·0011	·0000
Everitt	H	·0103	·006	·0005	·001	·0005	·001	·001	·0000

Section VI.—Synopsis.

1. Photographs of the diffraction pattern due to a heliometer objective with relatively long exposures have been secured which show certain features hitherto unsuspected by previous observers. The pattern roughly consists of two sets of parabolas (with a common axis, at right angles to the bright horizontal ray) branching in opposite directions. These features do not clearly appear in the relatively small region covered by the pictures of previous workers. Another important feature not mentioned by Everitt, who has made a detailed study of the diffraction figures, is that the fluctuations of intensity along the long bright ray that appears crossing the pattern decrease both relatively and absolutely at large angular deviations.

2. A new geometrical theory has been developed by means of which the light intensity at any point in the focal plane is considered as merely due to the interference of two or three light sources situated at definite points on the boundary. This is done by a simple transformation of the ordinary expression for the light vector taken as an integral over the surface of the aperture, into a line integral taken round the boundary, which latter is

finally reduced to two or three sources of proper phase and amplitude situated on the boundary of the aperture. This method greatly facilitates numerical computation of the light intensity for small as well as for very large angles of diffraction (the latter being almost impossible to obtain by the ordinary treatment), and gives results, which are for all practical purposes, as accurate as that obtained by elaborate mathematical formulae. It has also the additional advantage of enabling the form of the pattern to be readily determined for any desired area surrounding the focus, and the geometrical form thus deduced is in close agreement with that determined experimentally.

3. The writer has made preliminary observations on the diffraction figures due to apertures greater and less than a semicircle respectively, and has found that in these cases also the form of the pattern can be deduced from the above simple geometric considerations.

The investigation was carried out in the Palit Laboratory of Physics, and the writer wishes to thank Prof. C. V. Raman for his helpful interest during the progress of the work.

II.—Experiments with Mechanically-Played Violins.

By Prof. C. V. Raman.

(Plate II.)

CONTENTS.

SECTION I.—INTRODUCTION.

In the first volume (recently published *) of my monograph on the theory of the violin family of instruments, I have discussed on mechanical principles, the relation between the forces exerted by the bow and the steady vibration maintained by it, and the conditions under which the bow is capable of eliciting a sustained musical tone from the instrument. An experimental test of the results indicated by the theory on these points would obviously be of interest. Especially is this the case, as the analysis shows that the yielding of the bridge and the communication of energy from the strings through their supports into the instrument and thence into the air, play a very large part in determining the

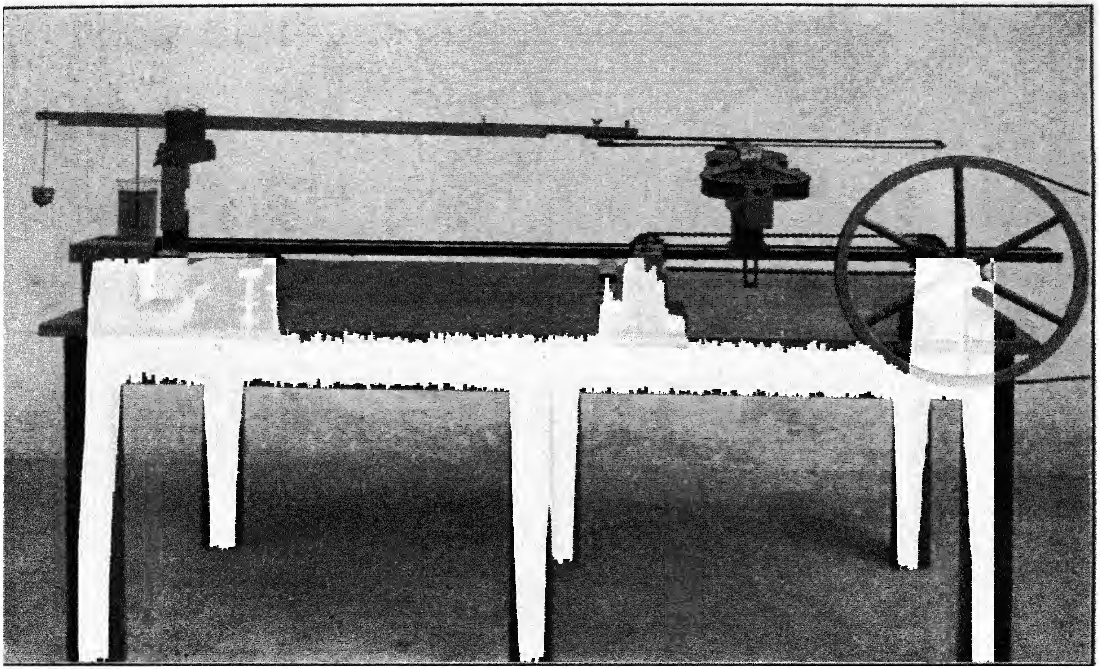
* Bulletin No. 15 of the Indian Association for the Cultivation of Science, 1918, pages 1-158.

magnitude of the forces required to be exerted by the bow. An experimental study of the mechanical conditions necessary for obtaining a steady musical tone could thus be expected not merely to throw light on the *modus operandi* of the bow but also to furnish valuable information regarding the instrument itself, its characteristics as a resonator and the emission of energy from it in various circumstances. Further, a study of the kind referred to could be expected also to furnish illustrations of the physical laws underlying the technique of the violinist and to put these laws on a precise quantitative basis. The experiments described in the present paper were undertaken with the objects referred to above, and the description of the results now given in these Proceedings is preliminary to a more exhaustive treatment of the subject which it is proposed to give in the second volume of my monograph under preparation for publication as a Bulletin of the Association.

SECTION II.—DESCRIPTION OF MECHANICAL PLAYER.

As the object of the work was to elucidate the theory of production of musical tone from instruments of the violin family, it was decided that the experimental conditions should approximate as closely as practicable to those obtaining in ordinary musical practice. The general principle accordingly held in view in designing the mechanical player was to imitate the technique of the violinist as closely as possible. There was also another reason for adopting this course. It is well known that the bowing of a stringed instrument so as to elicit a good musical tone is an art requiring much practice for its perfect accomplishment. The performance of the same task by purely mechanical appliances under such conditions as would permit of accurate measurements of the pressure and speed of bowing and the discrimination by ear of the effect of varying these factors obviously involves difficulties which it was thought would be best surmounted by imitating the violinist's handling of the bow as closely as the mechanical conditions would permit. A mechanical player designed on this general idea which has fulfilled the requirements of the work is illustrated in Plate II.

As can be seen from the photograph, a violin and a horse-hair bow of the ordinary type were used in the mechanical player.



A Mechanical Violin-player for Acoustical Experiments.

Instead, however, of moving the bow to and fro, it was found a much simpler matter from the mechanical point of view to keep the bow fixed and to move the violin to and fro with uniform speed. This was arranged by holding the violin lightly fixed in a wooden cradle, the points of support being the neck and the tail piece of the violin as in the ordinary playing of the instrument. The cradle was mounted on a brass slide which moved to and fro noiselessly on a well-oiled cast-iron track. The slide received the necessary movement forward and backward from a pin carried by a moving endless chain and working in a vertical slot carried by the slide. The chain was kept in motion by the rotation of one of the two hubs between which it was stretched, this hub being fixed on the same axis as the driving wheel seen in the Plate.*

The apparatus was driven by a belt running over a conical pulley which in its turn was driven by a belt passing over the pulley of a shunt-wound electric motor controlled by a rheostat which was allowed to run without any load except the apparatus. Using the rheostat and a Weston Electrical Tachometer, a very constant speed could be maintained during the experiments. Different speeds of motion of the slide carrying the violin were obtained by putting the driving belt of the apparatus on to different parts of the conical pulley, or by adjusting the rheostat.

The mounting of the bow required special attention in order to ensure satisfactory results. As is well known, the violinist in playing his instrument handles the bow in such manner that when it is applied with a light pressure, only a few hairs at the edge touch the string. The bow is held carefully balanced in the fingers of the right hand, the necessary increases or decreases in the pressure of bowing being brought about by increase or decrease of the leverage of the fingers. The suppleness of the wrist of the player and the flaccidity of the muscles of the fore-arm secures the necessary smoothness of touch. These features are carefully imitated in the mechanical player. The violin-bow is held fixed at the end of a wooden lath, an adjustment being provided so that fewer or more hairs of the bow may be made to touch the string

* The whole of the apparatus was improvised in the laboratory from such materials as were to hand. The slide and cast-iron track were parts of a disused optical bench. The chain and hubs were spare parts purchased from a cycle-dealer. The ball-bearing of the axle of the lever (referred to below) was also part of a cycle. The other fittings were made up in the workshop.

of the violin. The lath itself is balanced after the manner of a steelyard, the axis of the lever being mounted on ball-bearings so as to secure the necessary solidity combined with freedom of movement. The weight of the bow is balanced by a load hung freely near the end of the shorter arm of the lever. The axis of the lever can be raised or lowered to the proper height above the violin such that when the hairs of the bow touch the string, they are perfectly parallel to the cast-iron track along which the violin slides. This is of great importance in order to obtain steady bowing, as otherwise the bow would swing up and down with the movement of the violin along the track, and its inertia would result in a variation of the pressure exerted by it. Any residual oscillations of the bow due to the elasticity of the lever or imperfection in the adjustment referred to above are checked by the damping arrangement shown in the Plate. A wire with a number of horizontal disks attached to it at intervals is hung freely from the shorter arm of the lever and dips inside a beaker of water or light oil. This effectually prevents any rapid fluctuations in the pressure of the bow and ensures a smooth movement. The pressure exerted by the bow on the string can be varied by moving a rider along the longer arm of the lever which is graduated. An adjustment is provided by which the block carrying the axis of the lever can be moved by a screw perpendicular to the track, and the distance from the violin-bridge of the point at which the bow touches the string may thus be expeditiously altered.

It will be noticed that with the arrangements described above, the pressure exerted by the bow on the string of the violin would not be absolutely constant throughout, but would vary somewhat as the violin moves along its track from the point nearest to the point furthest from the axis of the lever. This is not however a serious difficulty as the lever is fairly long and the variation of pressure is thus not excessive. Further, the observations of the character of the tone are always made for a particular position and direction of movement of the violin and no ambiguity or error due to the cause referred to above arises.

The speed of bowing may be readily determined from the readings of the Electrical Tachometer or directly by noting on a stop-watch the time taken for a number of strokes to and fro of the violin on its track.

Section III.—Variation of Bowing Pressure with the Position of the Bowed Region.

One of the well-known resources of the violinist is to bring the bow nearer to or to remove it further away from the bridge of the violin, the extreme variation in the position of the bow being from about $\frac{1}{5}$th to about $\frac{1}{25}$th of the vibrating length of the string from the bridge. In a recent paper on "The Partial Tones of Bowed Stringed Instruments" published in the "Philosophical Magazine" (November 1919), I have discussed in some detail the changes in the amplitudes and phases of the various partials brought about by these changes in the position of the bowed region. In all the cases of musical interest within these limits, the mode of vibration of the string is practically the same as in the principal Helmholtzian type * in regard to the first three partial components, but differs from it in respect of the higher components to an extent depending on the removal of the bow from the bridge. The ratios of the amplitudes and the relative phases of the first three partials remain practically the same throughout the range, the actual amplitudes for a given speed of the bow varying inversely as the distance of the bowed point from the bridge. The amplitudes of the fourth, fifth and higher partials vary in a similar way with the position of the bowed point provided this is not too far from the bridge, but deviate from this law more and more as the bow is removed further and further from the bridge. The net effect of bringing the bow nearer the bridge (its speed remaining constant) is greatly to increase the intensity of the tone of the instrument, and to make it somewhat more brilliant in character, as is of course well known. Simultaneously with these changes, the pressure with which the bow is applied has to be increased. The mechanical player described above may be used to find experimentally the relation between the bowing pressure and the position of the bow under these conditions.

The graphs in Fig. 1 (thin lines) represent the results obtained with the player on the D-string of the violin. A few words of explanation are here necessary. As a finite region of the bow is in contact with the string, it is not possible to specify the position

* The principal Helmholtzian type is the mode of vibration in which the time-displacement graph of every point on the string is a simple two-step zig-zag.

of the bow by a single constant. Accordingly, the positions of the inner and outer edges of the region of contact were noted in the observations. The graph therefore shows two curves connecting the positions of the two edges of the bowed region with the

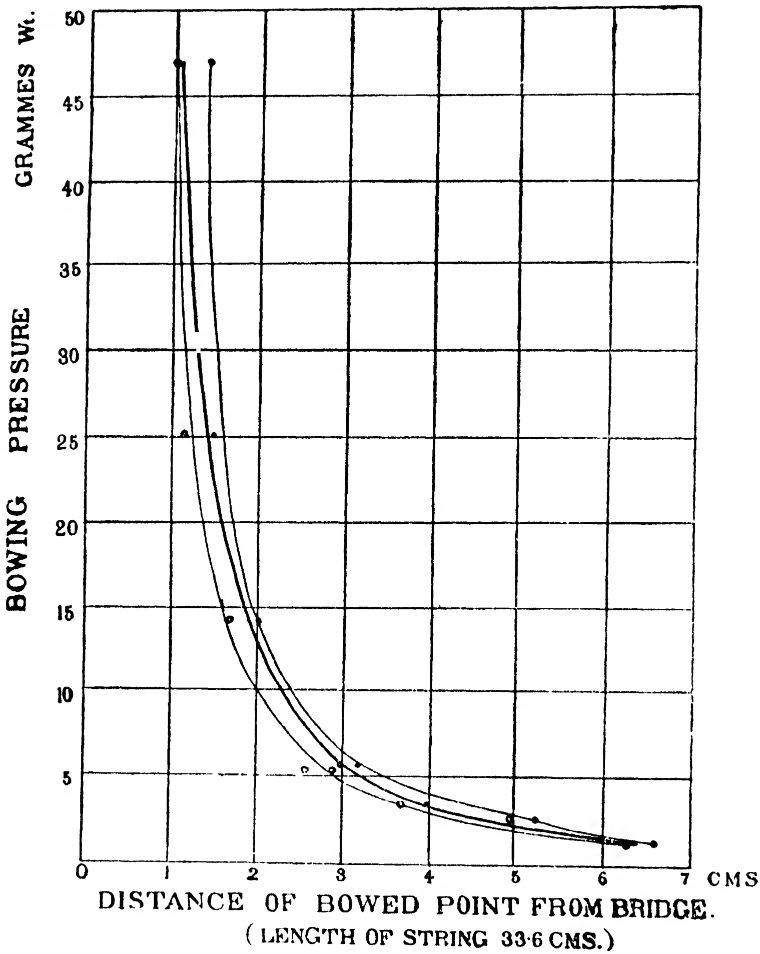


FIG. 1.

magnitude of the bowing pressure, The ordinates of the graphs represent the values of the *minimum* bowing pressure found necessary to elicit a full steady tone with pronounced fundamental. [For bowing pressures smaller than this minimum, the fundamental

falls off in intensity, and the prominent partial becomes its octave or twelfth. In certain cases, as for example near the wolf-note pitch, we get 'cyclical' or 'beating' tones.] It will be noticed from the graphs that the bowing pressure necessary increases with great rapidity when the bow is brought near the bridge.

The curve in Fig. 1 (heavily drawn) lying between the experimental graphs is a representation of the algebraic curve $x^2y=$ constant. (The constant was, of course, suitably chosen.) It will be noticed that the graph follows the trend of the experimental values quite closely. In other words, we may say that in the cases studied, the bowing pressure necessary varies practically in inverse proportion to the *square* of the distance of the bow from the bridge. It may be readily shown that this is the result to be expected from theory. In my monograph,* I have shown that the minimum bowing pressure P is given by the formula

$$P=\frac{P_A'-P_A}{\mu-\mu_A}$$

where P_A' is the maximum value at *any* epoch of the series

$$\sum_{n=1}^{n=\infty} k_n B_n \frac{\sin\left(\frac{2n\pi t}{T}+e_n+e_n'\right)}{\sin\frac{n\pi x_0}{L}},$$

and P_A is the value of the series at the epoch at which the bowed region of the string slips past the bow. μ is the statical coefficient of friction, and μ_A is the dynamical coefficient of friction during the epoch of slipping. B_1, B_2, etc. are the amplitudes of the partial vibrations of the string, k_1, k_2, etc. are numerical constants for the respective partials depending on the instrument, the mass, length and tension of the string, and x_0 is the distance of the bowed point from the end of the string. We have already seen that amplitudes B_n of the first few partials for a given speed of bowing vary in inverse proportion to the distance of the bow from the bridge, and their relative phases remain unaltered. In respect of these partials, the factor

$$1/\sin\frac{n\pi x_0}{L}$$

* Bulletin No. 15, pages 73 to 75.

also varies practically in inverse proportion to x_0, so long it is a small fraction of l.

To effect a simplification, we may proceed by ignoring the influence of all the partial vibrations except the first few, an assumption which is justifiable in the case under consideration, as by far the greater proportion of the energy of violin-tone is confined to the first few partials. Further, we may for simplicity, treat the difference $(\mu-\mu_A)$ between the statical and dynamical coefficients of friction as practically a constant quantity. This will not introduce serious error, provided the speed of the bow is not very small. For, if the slipping speed be fairly large, any changes in it due to change of the position of the bowed point would not seriously alter the dynamical coefficient of friction. On these simplifying assumptions, it will be seen from the formulæ given above that, within the limits considered, the minimum bowing pressure should vary in inverse proportion to the *square* of the distance of the bow from the bridge, exactly as found in experiment. This relation would, of course, cease to be valid when the bowed point is removed too far from the bridge or when the speed of the bow is very small.

SECTION IV.—RELATION BETWEEN BOWING SPEED AND BOWING PRESSURE.

The changing of the speed of the bow is another of the well-known resources of the violinist. The principal effect of this is to alter the intensity of tone. *Pari passu* with the change of speed of the bow, other things remaining the same, the violinist has to alter the pressure of the bow. The relation between these may be readily investigated with the mechanical player. The experimental results for the D-string and for a particular position of the bowed point are shown in Fig. 2.

The graph shows the following features: (1) for very small bowing speeds, the bowing pressure tends to a finite minimum value; (2) the increase of bowing pressure with speed is at first rather slow; (3) later, it is more rapid, the pressure necessary increasing roughly in proportion to speed, and for large amplitudes of vibration possibly even more than in proportion to the speed of the bow.

The foregoing results are, broadly speaking, in agreement with what might be expected on theoretical grounds.* This can be seen from the formula for the bowing pressure referred to in Section III. With increasing speed of the bow, the amplitudes B_n of the partial vibrations increase in proportion, so that if the difference $\mu - \mu_A$ between the statical and dynamical coefficients of friction be

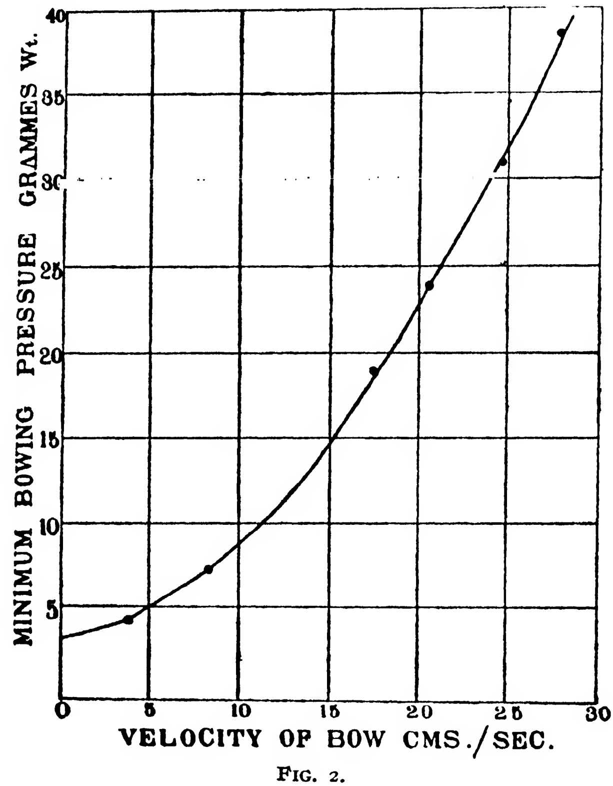


Fig. 2.

regarded as a constant, the bowing pressure necessary should vary directly as the speed of the bow. For very small speeds of the bow, however, it is not correct to take $\mu - \mu_A$ as constant, and it would be nearer the mark for such speeds to take $\mu - \mu_A$ as proportional to the velocity of slip, that is, as proportional to the speed of the bow. Thus, for very small speeds of the bow, the pressure neces-

* Bulletin No. 15, pages 151-153.

sary should be nearly independent of the speed, that is, should converge to a finite minimum speed. For larger speeds of the bow, it would be correct to take $\mu - \mu_A$ as constant and the bowing pressure should then vary proportionately with the speed. For very large speeds, the theory of small oscillations would no longer be applicable, and the quantities k_1, k_2, k_3, etc. might increase with the speed of the bow. For such large speeds, the bowing pressure necessary might increase more than in proportion to the speed of the bow.

A more precise discussion of the experimental results would be possible on the basis of quantitative data as to the manner in which the coefficient of friction between rosined horse-hair and catgut varies with the velocity of slip at different pressures.

SECTION V.—VARIATION OF BOWING PRESSURE WITH PITCH.

The pitch of violin tone depends on (1) the linear density of the bowed string, (2) its length, and (3) its tension, and may be varied by varying any one or other of these factors. In practice, the violinist varies the pitch by (1) altering the vibrating length by "stopping" down the string on the fingerboard, or (2) by passing from one string to another. The mechanical player may be used to investigate the dependence of bowing pressure upon pitch when the latter is varied in any of the ways that may be suggested. Obviously, the sequence of phenomena observed would not be exactly the same for the four strings of the violin as these are of different densities and tension, communicate their vibrations to the body of the instrument at different points of the bridge and also vibrate in considerably different planes relatively to the bridge and belly when excited by the bow in the usual way. In the experimental work now to be described, a particular string of the violin, e.g. the 4th or G-string, was used, and the pitch was varied as in the ordinary playing of the instrument by 'stopping' the string at different points. This was arranged by clamping the string down to the fingerboard, with a light but strong brass clamp shaped like an arch which could be put across the fingerboard, passed down upon it and then lightly fixed to it by two set-screws at the two ends. The inner face of the clamp was lined with leather to imitate the ball of the fingers of the violinist and to prevent damage to the strings.

A few remarks are here necessary. In actual practice, when the violinist stops down the string so as to elicit a note of higher pitch, he generally takes the bow up rather nearer the bridge so as to preserve the relationship between the vibrating length of the string and the distance of the bow from the bridge. Strictly speaking, this should also have been done in the present investigation. But as it would have been somewhat troublesome and involved the risk of errors in the adjustment of the position of the bow, it was decided to keep the bow in a fixed position somewhat close to the bridge and to find the relationship between the bowing pressure and the pitch of the string under these conditions. We have already seen that when the bow is fairly close to the bridge,

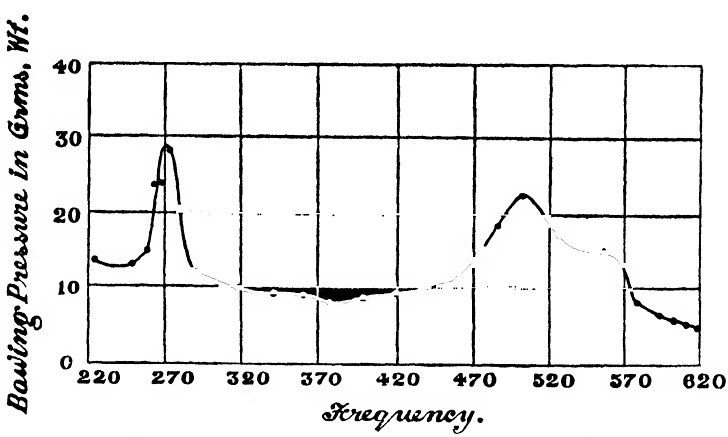


FIG. 3.—Relation between Bowing Pressure and Pitch (without Mute).

the bowing pressure necessary varies practically in inverse proportion to the square of the distance of the bow from the bridge. Accordingly, the effect of keeping the bow in a fixed position when the pitch is altered, instead of it bringing it nearer the bridge at each stage, is to decrease the bowing pressure necessary in a progressive and calculable ratio. This effect does not accordingly interfere with our observation of the characteristic changes of bowing pressure with pitch, which are connected with the changes in the forced vibration of the bridge and belly of the violin brought about by the change in the frequency of excitation.

The graph in Fig. 3 represents the relationship between bowing pressure and pitch within a part of the range of tone

of the violin which includes the first three of the natural frequencies of vibration of the body of the instrument. The particular violin used was of German make, marked copy of Antonius Straduarius, the bridge being of the usual Straduarius model. The experiments were made on the 4th or G-string, stopped down to various pitches. It will be noticed that the graph for the bowing pressure shows pronounced maxima and minima. There is a strong maximum at 270, another maximum between 470 and 520, and a distinct hump between 520 and 570. These maxima pretty nearly coincide in pitch with the first three maxima of intensity of the fundamental in the tone of the violin as estimated by ear, in other words with the frequencies of maximum resonance of the instrument to the gravest component of the force exerted on it by the vibrating string. The maximum lying between 470 and 520 is specially interesting as this region exhibits the well-known phenomenon of the 'wolf-note.' In the ascending part of this portion of the graph, and especially at and near the peak of the curve, it is found that when the pressure of the bow is less than the minimum required to elicit a steady tone with a well-sustained fundamental component, we get 'cyclical' or beating tones of the kind described and illustrated by me in previous papers.* The rapidity of the beats depends on the pitch of the tone which it is attempted to elicit. and also on the pressure and speed of the bow. A similar tendency to production of a 'wolf-note' though not so striking, is also manifested in the part of the graph between 520 and 570. The maximum in the region of 250 to 285 does not show a similar tendency, at any rate to any appreciable extent. It would appear that the gravest resonance of the violin chiefly involves a vigorous oscillation of the air within the belly of the instrument, but not so vigorous an oscillation of the bridge and belly as in the second and third natural modes of vibration which show the wolf-note phenomenon. Further evidence on this point is furnished by experiments on the effect of putting a load or mute on the bridge of the violin as will be referred to in the following section.

The formula for the bowing pressure quoted on page 25 enables the variation of bowing pressure shown in Fig. 3 to be explained. In the series

* Bulletin No. 15, also Phil. Mag. Oct. 1916.

$$\sum_{n=1}^{n=\infty} K_n B_n \frac{\sin\left(\frac{2n\pi t}{T} + e_n + e_n'\right)}{\sin \frac{n\pi x_0}{L}},$$

of which the graph practically determines the bowing pressure required, the B_n's stand for the amplitudes of the different partial vibrations of the string, and the K_n's are quantities which are practically proportional to the corresponding partial components of the forced vibration of the bridge transverse to the string at its extremity. In the case of a string bowed near the end, $B_n / \sin \frac{n\pi x_0}{L}$ varies nearly as $1/n^3$ and thus decreases rapidly as n increases. Further, in the part of the range of violin-tone covered by the graph in Fig. 3, the fundamental component of the vibration of the bridge should obviously be well marked, and K_1 would therefore be of the same order of quantities as K_2, K_3, etc., or even larger. Hence the value of the series given above would be principally determined by its leading term proportional to K_1, and the variation of bowing pressure with pitch would practically follow the fluctuations of K_1, in other words would follow the variations in the amplitude of the fundamental component in the forced vibrations of the bridge and pass through a series of maximum values at the successive frequencies of resonance of the instrument. This is practically what is shown by the experimental results for the bowing pressure appearing in Fig. 3, and the graph gives us an idea of the *sharpness* of the resonance of the instrument at each of the frequencies referred to. It must be remembered, however, that the bowing pressure required is also influenced by the terms proportional to K_2, K_3, etc., that is by the resonance of the instrument to the second, third and higher partial components of the vibration, and some evidence of this also appears in the graph in Fig. 3. For instance, though the first resonance of the instrument was actually found to be at a pitch of 284, the peak of the curve for the bowing pressure is at about 270 as can be seen from Fig. 3. This appears to be a consequence of the fact that the course of the curve is to some extent modified by the resonance of the *octave*. It is obvious that corresponding to the resonance of the instrument to the fundamental tone in the range of pitch from 470 to 570, the octave should be strongly reinforced when the pitch lies

within the range 235 to 285; hence the peak of the curve for the bowing pressure instead of being at 284 is actually shifted towards a lower frequency (270) as is seen from Fig. 3.

Obviously, the experiments described in this section may be extended in various directions. The curves for the three other strings of the violin, and especially over the whole of the possible range of pitch of the tone of the instrument, and the differences between the curves obtained with the different strings would deserve investigation. The differences between different violins could obviously be studied by this method, and the constants K_1 K_2, etc., for any particular violin and string and for various pitches may be found experimentally by study of the free and forced oscillations of the bridge, and used for a theoretical calculation and comparison with experiment of the bowing pressure required for exciting the tone of the instrument.

SECTION VI.—EFFECT OF MUTING ON BOWING PRESSURE.

Perhaps the best illustration of the close relation existing between the forces required to be exerted by the bow and the

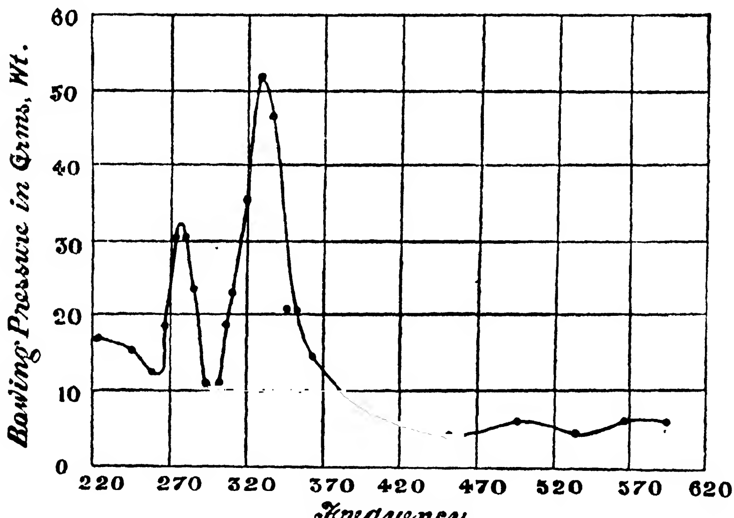


FIG. 4.—Relation between Bowing Pressure and Pitch (with Mute).

communication of vibrations from the string to the bridge and belly of the instrument and thence to the air, is furnished by the

effect of putting a mute on the bridge on the bowing pressure required for eliciting a musical tone. Very striking results on this point may be obtained with the aid of the mechanical player. Fig. 4 illustrates the relation between bowing pitch and bowing pressure observed experimentally, the G-string of the violin being used as in Fig. 3, and the results shown in Fig. 3 and Fig. 4 being obtained under the same conditions except that in the latter case, a brass mute weighing 12·4 grammes was clamped to the bridge, while in the former case, the bridge was unmuted. The great difference between the two cases is obvious, and the change in the form of the graph for the bowing pressure shows a very close analogy with the change in the character of the forced vibration of the instrument and the intensity of the tone of the violin over the whole range of pitch produced by application of the mute.

In view of the discussion of the graph for the bowing pressure contained in the preceding section, and the theoretical treatment of the effect of muting already given by me in previous papers* it is perhaps not necessary to enter here into a detailed examination of the subject, and it may suffice briefly to draw attention to some of the features appearing in the graph in Fig. 4. It will be noticed that there is a peak in the curve at a frequency of about 280. This is the pitch of the first resonance of the instrument, the fundamental component of the vibration being re-inforced, and in this case, the position of the peak in the curve for the bowing pressure is not appreciably influenced by the resonance of the octave as in Fig. 3. It is clear that the pitch of the first resonance of the instrument is hardly influenced at all by the application of a load of 12·4 grammes to the bridge, and the bowing pressure required at the first peak of the curve is nearly the same as in Fig. 3. The first natural mode of vibration of the violin does not therefore appear to involve any very large vibration of the bridge. Following the peak at 280, we have in fig. 4 a very high peak near 330 which is the pitch of the 'wolf-note' as lowered by the mute of 12·4 grammes. The great lowering of pitch (from 490 to 330) shows that the second mode of vibration of the body of the violin involves a very large vibration of the

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*** Phil. Mag. October 1916, Nature October 1917, Phil. Mag. June 1918, and Bulletin No. 15, pages 143 to 151.**

bridge, and the enormous increase of bowing pressure at the peak of the curve is also noteworthy. This can no doubt be explained on dynamical principles as due to the very greatly increased amplitude of the forced vibration of the bridge due to the loading. At higher pitches, the bowing pressure necessary falls off very rapidly, though one or two minor maxima (due to the resonance of the instrument in its higher modes as altered by the loading) are also obtained. The tone of the instrument in the higher ranges of pitch when muted is extremely feeble.

Further investigations which are worthy of being carried out would be the study of the effect of gradually increasing the mass of the mute on the graph for the bowing pressure, and also of putting the mute at different places on the bridge. In view of what has been stated above, it is clear that we may expect the changes in the form of the graph to follow closely the changes in the pitch of resonance of the instrument in its various modes produced by the loading.

SECTION VII.—OTHER APPLICATIONS OF THE MECHANICAL PLAYER.

The investigations described in the preceding sections may be extended in various directions. Some indications have already been given on these points, and it will suffice here to suggest some of the other possible applications of the mechanical player. As the instrument affords a means of bowing the violin at precisely measurable speeds and pressures, it furnishes a means by which the intensity of violin-tone and its variations with pitch may be quantitatively determined and compared with the indications of mathematical theory. Various questions, such as for instance the effect of heavier or lighter stringing, the effect of varying the pressure, speed, and the width of the region of contact of the bow, and the position of the bowed region on the tone-quality of the violin may be quantitatively studied with a degree of accuracy that cannot be approached in manual playing with its undetermined conditions. Further, the study of the tone-intensity, of the bowing pressure curves, and of the vibration-curves of bridge and belly of the instrument under quantitative conditions made possible by mechanical playing may be expected speedily to clear up various structural problems relating to the construction of the

violin, e.g. the effect of the peculiar form of the Stradinarius bridge, the influence of its position, the function of the sound-post and base-bar, the shape of the air-holes, the thickness, curvature and shape of the elastic plates composing the violin, and the influence of various kinds of varnish. The dynamical specification of the constants determining the tone-quality of any violin over the whole range of pitch may be regarded as one of the aims towards which these investigations are directed.

Section VIII.—Synopsis.

The paper describes the construction of a mechanical violin-player intended for study of the acoustics of the instrument, and some of the investigations in which it has been applied. The principal feature in the player which is worthy of notice is that the conditions obtaining in ordinary musical practice are imitated with all the fidelity possible in mechanical playing, and the results obtained with it may therefore be confidently regarded as applicable under the ordinary conditions of manual playing. The following is a summary of the results obtained in the four investigations described in the present paper: (1) *Effect of the position of the bowed region on the bowing pressure*: it is shown that provided the speed of the bow is not too small, the bowing pressure necessary within the ordinary musical range of bowing varies inversely as the *square* of the distance of the bow from the bridge. (2) *Relation between bowing speed and bowing pressure*: it is shown that for very small bowing speeds, the bowing pressure necessary tends to a finite minimum value, and the increase of bowing pressure with speed is at first rather slow, but later becomes more rapid. (3) *Variation of bowing pressure with pitch*: the graph for the bowing pressure for different frequencies shows a series of maxima which approximately coincide in position with the frequencies of resonance of the instrument. (4) *Effect of muting on bowing pressure*: it is found that the mute produces profound alterations in the form of the graph. The bowing pressure necessary is increased in the lower parts of the scale and decreased in the higher parts of the scale. The peaks in the graph shift towards the lower frequencies in consequence of the alteration in the natural frequencies of resonance of the violin produced by the loading, and the change in the form of the graph is closely

analogous to the change of the intensity of the fundamental tone of the instrument produced by the muting.

Some further possible applications of the mechanical player are also indicated in the paper.

III. Mechanical Illustration of the Theory of Large Oscillations and of Combinational Tones.

By Bhabo Nath Banerji, M.Sc., Assistant Professor of Physics in the Calcutta University.

(Plate III.)

CONTENTS.

Section I.—Introduction.

It is well known that the principle of superposition of small motions is valid only for systems with infinitesimal amplitudes of vibration, and that the cases in which it fails owing to the finiteness of the amplitudes may often rise to considerable importance in acoustics. Referring to the principle of superposition Helmholtz remarks: "There are certain phenomena which result from the fact that this law does not hold with perfect exactness for vibrations of elastic bodies which though almost always very small are far from being infinitesimally small. For infinitesimally small motions, the moving forces excited by the mutual displacement of the particles of the oscillating medium are simply proportional to these displacements, but as soon as these vibrations become large the higher powers of the displacements begin to

sensibly influence the motion." On this basis, Helmholtz has founded an explanation of the presence in certain cases of harmonics in the tones emitted by simple vibrators and of the production of combinational tones within the human ear.

The experimental illustration of Helmholtz's theory is a matter of great interest, and has attracted the attention of many investigators. Amongst the most recent work on the subject may be specially mentioned, the contributions of E. Waetzmann and W. Moser.* The production of combinational vibrations of strings by the simultaneous action of two simple harmonic forces varying the tension has also been described and illustrated by Professor C. V. Raman.† The present work taken up at the suggestion of Professor C. V. Raman proceeds on entirely different lines and is intended to furnish simple experimental and visible illustrations of the theory of large oscillations and of the production of combinational tones.

SECTION II.—DESCRIPTION OF APPARATUS.

The apparatus used was originally designed with an entirely different object, but has proved very well suited for the present purpose. It was at first intended to be used as an inharmonic synthesiser. The construction will be clear from Figs. (1) and (2) in Plate III. A wooden swinging arm is pivoted round an axle carrying ball-bearings so as to minimise friction (a bicycle-hub serves the purpose admirably). The axle is vertically fixed to a solid table, so that the swinging arm is free to move in a horizontal plane. (See Fig. 1 in Plate III.) The motion of the arm is checked and controlled by two strong steel springs tightly connected to it on either side, the other two ends of the springs being connected (subject to adjustment of tension) with two supports fixed to the table. The swinging arm carries a number of hooks from which a row of pendulums can be suspended (each by a pair of strings from consecutive hooks). When the pendulums are allowed to oscillate, they singly or jointly force a small oscillation of the

* Science Abstracts, Sec. A, May 1919.

† Bulletin of the Association, No. 11, 1914, and Physical Review, January, 1915.

Fig. 1

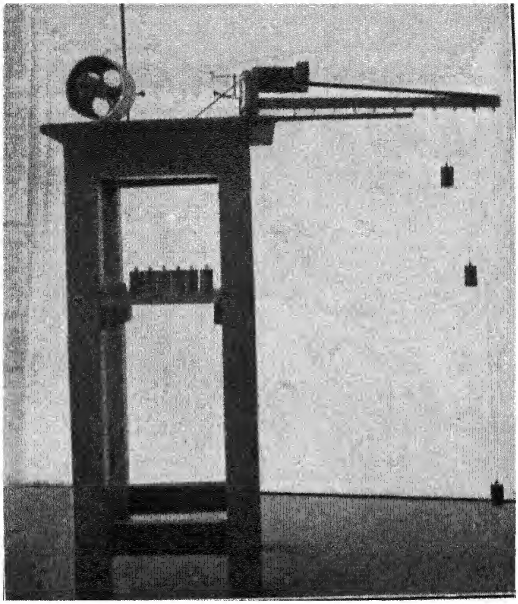
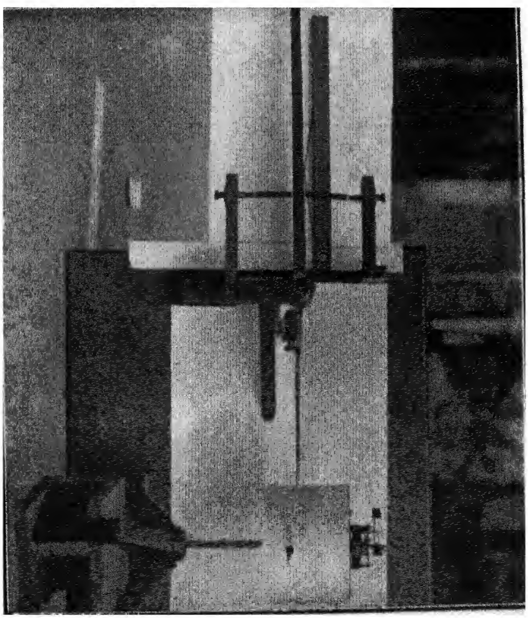


Fig. 2



Apparatus for Illustration of the Theory of Large Oscillations and Combinational Tones.

swinging arm. An arrangement is provided by which the oscillation of the arm may be magnified and recorded on paper if so desired. (This is not an essential part of the apparatus and may be left disconnected unless the records are required.) For this purpose the arm carries a vertical projecting pin which by working in a slot carried at the end of a long aluminium pen controls its movements. (See Fig. 2 in Plate III.) This pen is pivoted vertically at a small distance from the pin, the distance between pin and pivot being capable of adjustment with the aid of a screw. The pen carries a small reservoir of ink at the end—the flow of ink from the nib-point being made continuous by connecting the reservoir to the nib-point by a piece of cotton. This nib-point rests on a drum which can revolve about a horizontal axis by a clockwork arrangement and is covered by a sheet of paper.

The swinging arm in the apparatus as described above will be referred to as the symmetric vibrator in what follows—symmetric because the force of restitution for any displacement on either side of the position of equilibrium is the same. The swinging arm as controlled by the spring has a natural frequency of its own, its vibration when started being rather rapidly damped out on account of friction in the ball-bearings.

In order to make the vibrator asymmetric when desired, the following mode of attachment was resorted to. In addition to the two steel springs, two steel wires meeting at a very obtuse angle control the movement of the swinging arm. (See Fig. 2 in Plate III.) This attachment causes the force of restitution acting on the arm to be unsymmetrical. The wires tend strongly to resist stretching and consequently the restoring force on one side of the equilibrium position for the same displacement is not the same as that on the other side.

The vibrator in either case (symmetric or asymmetric) has a natural frequency determined by its inertia and the strength of the springs. A pendulum hanging from the vibrator when set in periodic oscillation exerts a periodic force on the latter—the magnitude of the force depending on the weight of the pendulum bob and its maximum angular displacement, and the period of the force is the same as the period of oscillation of the pendulum. If two or more simple harmonic forces of different magnitudes and periods be

required to be superposed on the vibrator, the same number of pendulums (which may be called the forcing pendulums) should be used, the magnitude being adjusted by the weight of the bob and the period by the length of supension. Moreover if two forces at different phases are to be superposed on the vibrator, this can be done by timing the starting of the two forcing pendulums to correspond to the required phase difference. The swinging arm in all cases will vibrate under the joint action of all the forces acting simultaneously upon it. This mode of vibration of the vibrator can be recorded by the movement of the inking pen on the horizontally rotating drum covered with white paper, the ink traces showing the time-displacement curve of the vibrator.

From the vibrator an exploring pendulum can be suspended and will then act as a resonator, for if in the resultant oscillation of the vibrator a simple harmonic component synchronous with the periodic time of the exploring pendulum be present, the latter will be set in vigorous oscillation. The sharpness of this resonance will depend on damping and hence the exploring pendulum should be fairly heavy. This pendulum while showing its resonance effect reacts on the swinging arm being connected to the latter in the same way as the forcing pendulums. This mode of vibration of the swinging arm in the presence of the resonators will be illustrated in the course of the paper.

It will be noticed that the pendulum method of illustration bears an analogy to the recent work of Professor. E. H. Barton and Miss Browning on forced oscillations described in the "Philosophical Magazine" (1918 and 1919).

SECTION III.—SINGLE FORCING.

I started a forcing pendulum with a small amplitude and recorded the ink trace of the vibrator.

FIG. (3).


FIG. (4).


FIG. (3) represents the vibration of the symmetric vibrator when a simple harmonic force of small intensity acts on it. The force is small, for the angular motion of the forcing pendulum on

which the former depends is very small. Here the curve is a simple sine curve; the motion being small, the first power of the displacement determines it.

FIG. (4) represents the vibration of the asymmetric vibrator under the action of simple harmonic force of small intensity. The asymmetry in the vibration is shown by the long flat portion of the curve which corresponds to very small movement of the vibrator against the wire attachment on one side, whereas on the other side the amplitude is greater though the curve is nearly flat in consequence of the higher powers of displacement determining the motion.

I then started the forcing pendulum with a large amplitude and obtained the following ink traces of the vibrator.

FIG. (5).

FIG. (6).

FIG. (5) represents the large oscillation of the pendulum as perfectly traced by the symmetric vibrator. The curve is not a simple sine curve and unlike fig. (3) the crests and troughs are flattened out, the remaining portions becoming comparatively steeper. The shape of the curve indicates the complexity of the vibration caused by the presence of the higher harmonic components in the resultant vibration, and the perfect symmetry on both sides indicates the absence of the even harmonics.

FIG. (6) represents the vibration of the asymmetric vibrator under the large amplitude of the forcing pendulum. It is very nearly the same as fig. (4) except that the asymmetry is more marked, and the nearly flat tops of the former curves have become flatter still, this portion being traced in the same circumstances as fig. (5).

For an analysis of the above vibrations, I started the forcing pendulum with a considerable amplitude and determined its period by means of a stop watch. I then made the length of the exploring pendulum to start with a little longer than that of the forcing pendulum and observed the effect of the latter on the former for all lengths right up to the shortest possible.

With the symmetrical vibrator it is found that the lengths of the exploring pendulum for which there is maximum resonance correspond with periodic times which are respectively the same and as one-third the period of the forcing pendulum. For all intermediate lengths the exploring pendulum takes up no regular oscillation except for lengths which are nearly equal to the resonating lengths. In the latter cases it is observed that the exploring pendulum takes up a periodic oscillation with varying amplitudes corresponding to beats, the frequency of which diminishes as the resonating length is approached. This phenomenon of beats helps the final adjustment of pendulum length corresponding to a resonance frequency. The periodic time for the latter being determined after every such adjustment confirms the presence of the fundamental as well as the third harmonic. It is observed that the length of the exploring pendulum resonating to the fundamental is longer than the forcing pendulum. The explanation of this lies in the fact that the periodic time of the pendulum with a large amplitude corresponds to a slightly longer pendulum with a smaller amplitude. The angle of swing of the resonant pendulum is much greater for the third harmonic than for the fundamental. It is also observed that the exploring pendulum set to give the octave shows no resonance. When the forcing pendulum is started with a lesser amplitude there is a very considerable decrease in the intensity of resonance, the third harmonic dying out more rapidly than the fundamental.

The observations described above are fully explained when we consider the nature of the forces exerted on the vibrator by the swinging pendulum. For small oscillations, the pendulum exerts a simple harmonic force synchronous with its own oscillation on the swinging arm from which it is suspended. But when the oscillation is large, this reaction includes also a third harmonic as may be readily shown.

The differential equation of motion of pendulum is

$$Ml\frac{d^2\theta}{dt^2} = -Mg\sin\theta$$

If α be the semivertical angle of swing of the pendulum the first approximate solution gives

$$\theta = \alpha\cos wt\ ,\ \text{where } w = \sqrt{\frac{g}{l}}$$

the second approximate solution becomes

$$\theta = a'\cos w't - \frac{a'^3}{192}\cos 3\, w't$$

where $w' = w\left(1 - \frac{a^2}{16}\right)$ and a' is slightly different from a.

The horizontal reaction on the vibrator which is

$$Mg \cos\theta \ \sin\theta \text{ or } Mg\left(\theta - \tfrac{2}{3}\,\theta^3\right)$$

approximately becomes after substitution

$$Mg \ \left\{ a'\cos w't - \tfrac{1}{6}\, a'^3 \cos 3\, w't \right\} \text{ approximately.}$$

If n be the natural frequency of the vibrator, its oscillation will be represented by

$$\ddot{y} + n^2 y = a'\cos w't - \tfrac{1}{6}\, a'^3 \cos 3\, w't$$

the constant factors depending on Mg and the inertia of the vibrator being omitted from the coefficients of the force terms in the equation. The solution then becomes

$$y = \frac{a'}{n^2 - w'^2}\cos w't - \tfrac{1}{6}\,\frac{a'^3}{n^2 - 9\,w'^2}\cos 3\, w't$$

It will be noticed from the above equations as well as by the mechanical analysis that the complexity of the vibration in fig. (5) is due to the presence of the third harmonic, the symmetry of the curve being explained by the absence of the even harmonics. The two resonances obtained are due to the corresponding forces exerted on the vibrator by the large oscillations of the forcing pendulum. The angular swing of the resonating pendulum corresponding to the third harmonic is large; for the forced motion of the vibrator having this frequency is proportional to the factor $\frac{1}{n^2 - 9w'^2}$, and is thus great, the natural frequency n of the vibrator being in the actual experiment nearly equal to $3w'$ the frequency of the resonant pendulum. Then again the intensities of resonance of the primary and the third harmonic depend respectively on a' and a'^3 which explains the phenomena that with small values of a' the third harmonic becomes very small, being dependent on a'^3 which

will then be negligibly small. It will be noticed that the explanation of the presence of the third harmonic in the motion given above does not involve any assumption that the springs controlling the motion of the vibrator deviate appreciably from Hooke's Law. Such a deviation, if it did exist however, would produce results practically analogous to those described above.

With the asymmetric vibrator it is found that the lengths of the exploring pendulum for which there is maximum resonance corresponds with periodic times which are (1) equal; (2) one-half; and (3) one-third of the period of the forcing pendulum. The angle of swing for the fundamental resonance is small, the length of the exploring pendulum being slightly greater than that of the forcing pendulum as remarked before. The second and third harmonics decrease in intensity when the amplitude of the forcing pendulum is decreased. In the case of the third harmonic the decrease is very marked, but the second harmonic persists even with diminished amplitudes. The presence of the third harmonic needs no further explanation in view of what has been stated above. In forming the approximate differential equation of motion for the vibrator for the present case, the force corresponding to the third harmonic may be omitted for simplicity. The equation then becomes

$$\ddot{y} + n^2 y + \beta y^2 = a \cos wt$$

where β is the constant of asymmetry.

A solution to the above is

$$y = k + \frac{a}{n^2 - w^2} \cos wt - \frac{\beta}{2(n^2 - 4w^2)} \left(\frac{a}{n^2 - w^2} \right)^2 \cos 2wt$$

The asymmetric factor β is involved in the amplitude coefficient of the second harmonic and hence the persistence of the latter though with smaller intensity for lesser forcing as represented by the presence of a^2 in the coefficient. That the intensities of resonance depend on the natural frequency of the vibrator is also evident from the solution.

The following curves illustrate the modification of the vibration of the vibrator in the presence of the resonator, the vibration curves without the resonators having been reproduced before. In getting the records, the forcing pendulum is started with a big amplitude and the exploring pendulum adjusted to give the particu-

lar resonance is allowed to hang undisturbed. This second pendulum takes up the oscillation by resonance and the records of the vibration are then obtained.

FIG. (7)

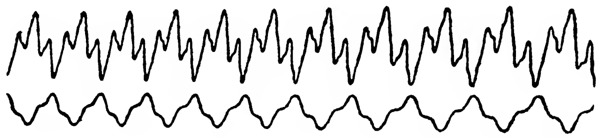


FIG. (6)

FIG. (8)

FIG. (9)

FIG. (10)

FIG. (7) represents the vibration of the vibrator in the presence of the resonator which corresponds to the third harmonic feebly present in the forcing pendulum. It will be noticed that the presence of the resonant pendulum results in an enormous magnification of the third harmonic in the resultant motion of the vibrator. FIGS. (8), (9) and (10) represent the vibrations of the asymmetric vibrator—fig. (8) is the vibration of the vibrator in the presence of a resonator which is the octave of the force acting; fig. (9) is in the presence of a resonator which corresponds to the third harmonic; fig. (10) is in the presence of resonators corresponding to the octave and the twelfth combined. It is evident from these figures that the effect of the resonant pendulums is very greatly to increase the amplitudes of the upper partials in the resultant vibration.

SECTION IV.—DOUBLE FORCING.

I next pass on to the case of two periodic forces acting on the vibrator simultaneously. For this purpose another pendulum is suspended from the vibrator just by the side of the first forcing pendulum. The two forcing pendulums are then vigorously started. With the symmetrical vibrator it is found that the exploring pendulum responds to primaries and twelfths of each of the forces acting. In fact, one or other of the forces acting singly gives all the resonances obtained, all observations being quite similar to those noticed before in the case of single

forcing. No resonance is observed when the exploring pendulum is adjusted to correspond to either the sum or the difference of the frequencies of the two forcings. The vibrator of course in the presence of the two forces takes up a vibration as if the algebraic sum of the two forces act singly on it.

A special case of the resultant vibration when the periods of the two primary forces acting are nearly equal is represented by the following ink trace of the vibrator. The phenomena of beats which the algebraic sum of the two components would give, the frequency of the beats being the difference of the frequencies of the two components, is clearly illustrated in Fig. (11).

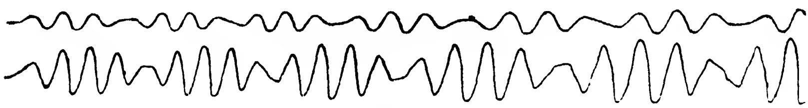


FIG. (11).

Having next made the vibrator asymmetric I started the two forcing pendulums vigorously and obtained resonance of the exploring pendulum in the cases shown in the following table. The choice of the two forcing pendulums was such that the sum and difference of their frequencies were not the same as any of the higher harmonics of either of the forces.

TABLE.

Period and frequency of forcing pendulums.	Resonating pendulum.		Relation.	Remarks.
	Period.	Frequency.		
$T_1 = 2{\cdot}08$ or $p = {\cdot}48$ $T_2 = 0{\cdot}9$ or $q = 1{\cdot}11$	2·08	·48	Primary.	Corresponding to p
	0·90	1·11	Primary.	,, to q
	1·04	·96	Octave.	,, to $2p$
	0·45	2·22	Octave.	,, to $2q$
	0·69	1·44	Twelfth	,, to $3p$
	0·63	1·59	Summational.	,, to $p+q$
	1·60	·63	Differential.	,, to $p \sim q$

The differential equation of motion for the above case is represented as

$$\ddot{y} + n^2 y + \beta y^2 = f \cos pt + g \cos qt$$

where p and q are the frequencies of the forcing pendulums, f and g the intensities of the respective forces acting. In the above equation the small forces of frequency $3p$ and $3q$ present in the corresponding large amplitude forcing pendulum have been left out of the equation for simplicity, the present intention being the illustration of the theory of combinationals. Hence a solution to the above approximate equation will not contain the corresponding terms, though they are actually present.

The solution of the above equation is of the form

$$y = K + A \cos pt + B \cos qt + C \cos 2pt + D \cos 2qt$$
$$+ E \cos \sigma t + F \cos \delta t$$

where $\sigma = p + q$ and $\delta = p - q$.

The results shown in the table are thus explained. The combinationals of the first order where σ represents the summational and δ the differential have for their intensity coefficients

$$E = \frac{\beta}{n^2 - \sigma^2} \cdot \frac{f}{n^2 - p^2} \cdot \frac{g}{n^2 - q^2}$$

$$F = \frac{\beta}{n^2 - \delta^2} \cdot \frac{f}{n^2 - p^2} \cdot \frac{g}{n^2 - q^2}.$$

The above equations explain conclusively the facts that are actually observed in the experiment. An increase in β the factor of asymmetry increases the combinationals. The alteration in the constant of asymmetry is mechanically illustrated by an alteration in the diameter or tension of the wires controlling the motion of the vibrator. An increase in f or g the magnitude of the primary forces acting increases the intensity of the combinationals. This is an illustration of the well-known fact that for the production of the combinational tones the generating tones must be loud and well sustained. Then again the factor $n^2 - \sigma^2$ or $n^2 - \delta^2$ indicates that the proximity of the frequency of the tone to the natural frequency of the vibrator increases that tone considerably. With my apparatus the summational approaches the above condition and hence its comparatively stronger resonance than with the differential.

It was also found that the two forcing pendulums with the exploring pendulum resonating to a combinational formed a system of which any two when vigorously started produced resonant oscillation in the third. For if a and b be the frequencies of the forcing pendulums and c that corresponding to the summational $a+b=c$ and

$$c-a=b$$
$$c-b=a$$

so that if c and a or c and b are started the respective resonances will be a or b corresponding to the differentials as the equations will show.

The following curves show the mode of vibration of the vibrator in the asymmetric system.

FIG. (12).

FIG. (12) is the vibration curve of the vibrator under double forcing and in the absence of any resonator. It also illustrates the phenomena of asymmetric beats.

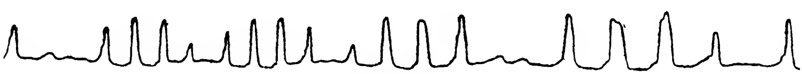


FIG. (13).

FIG. (13) represents the modified vibration curve of the vibrator in the presence of a resonator tuned to the summational. It will be noticed that the component motion corresponding to the summational frequency becomes markedly more prominent and obvious to inspection.

The asymmetric vibrator together with the resonators may roughly be taken as the mechanical representation of the human ear, the drum skin being compared to the asymmetric vibrator and the resonating chords of the basilar membrane to the exploring pendulums. From what has been said before, it will be quite clear that when two loud sources of sound affect the drum skin of the ear, the chords of the basilar membrane corresponding to the

harmonics and combinational tones are excited and the corresponding sensations are produced.

The vibration of the drum skin is not the simple sum of the motion due to the two primary disturbances acting separately but considerably modified because of its asymmetric configuration and also its connection to the inner vibrating parts of the ear including the basilar membrane. We have seen in the foregoing that the motion of the asymmetric vibrator under the joint action of two periodic disturbances may be greatly modified by the presence in connection with it of resonators tuned to the frequency of the combinational vibration, the effect being to magnify the amplitude of this part of the disturbance. The experiments thus also illustrate Helmholtz's remark that even where the production of the combinational tones occurs within the ear, the association of the ear with a suitable resonant cavity connected with it may tend to reinforce the combinational tones. Further, it is also not impossible that a similar reinforcement of the combinationals may occur within the ear itself in consequence of the connection with the drum skin of the internal parts capable of vibration and having free periods of their own. The natural frequency of the drum skin unlike the mechanical analogue is small in comparison with the frequency of the audible disturbances affecting it, and hence the stronger production of the differentials than the summationals in the human ear unlike the mechanical analogue where the summational is the stronger.

Section V.—Synposis.

The paper gives a short account of experiments illustrating the production of harmonics and combinational tones in systems having finite amplitudes of vibration. The apparatus consists of a swinging arm pivoted round a vertical axle and free to oscillate in a horizontal plane; the forces controlling this oscillation may be made either symmetric or asymmetric as desired. From the swinging arm a number of pendulums are suspended which can be set in oscillation. The behaviour of the pendulums can be observed, and provision is made by which the motion of the swinging arm due to their reactions may be recorded if desired, in the form of a time-displacement graph.

The apparatus has been used to demonstrate the following :—

(1) The presence of the third harmonic in the oscillations forced by a pendulum swinging through a large amplitude.

(2) The presence of the second harmonic in the oscillation of an asymmetric vibrator subject to a simple-harmonically varying force.

(3) The influence of a resonator of the appropriate frequency attached to the vibrator in either case in magnifying those components of its motion.

(4) The production of beats under double forcing of symmetric and asymmetric systems.

(5) The production of combinational oscillations of an asymmetric vibrator under double forcing, and the influence of the free period of the oscillator on the magnitude of the components of combinational frequencies in its motion.

(6) Thei nfluence of resonators tuned to the combinational frequencies upon the motion of the asymmetric system when connected with it.

In conclusion, the author wishes to express his cordial thanks to Prof. C. V. Raman for the facilities put at his disposal and for constant interest and encouragement in his research.

IV. Some Phenomena of Laminar Diffraction observed with Mica.

By Phanindra Nath Ghosh, M.A., Lecturer on Optics in the University of Calcutta.

(Plate IV.)

CONTENTS.

Section I.—Introduction.

In a paper recently contributed to the Proceedings of the Royal Society,* the author has described and explained the interesting phenomena observed when a sheet of mica is examined by the Foucault test or in the Toepler "Schlieren" apparatus as it is otherwise called. Certain lines or "striae" on the surface of the mica appear luminous and beautifully coloured, the colour depending on the angle at which the mica is held to the light incident on it in the apparatus. It was shown that the striae are the boundaries between regions of the mica having slightly different thicknesses, and it was pointed out in the paper that the colour of a stria as observed in the Foucault test is complementary

* "On the Colours of the Striae in Mica." Proc. Roy. Soc., 1919, Vol. 96, pp. 257-266.

to the colour of the central fringe in the laminary diffraction-pattern produced by it and observed in rear of the sheet of mica when plane waves are allowed to traverse it. Attempts were made to reproduce the colours observed in the Foucault test by using glass plates with artificially prepared laminar boundaries, obtained by etching out a very thin layer over part of the surface of the glass with dilute hydrofluoric acid. These attempts were not very successful, and this failure was ascribed in the paper to a want of sufficient abruptness or sharpness in the diffracting edges thus prepared. On the ordinary elementary theory, a sharp laminar boundary should give a diffraction-pattern which is more or less exactly symmetrical in configuration about the central fringe.* In practice, etched glass plates give diffraction-patterns which are markedly asymmetrical,† the fringes on one side of the centre being much brighter than those on the other. Careful observations showed that even in the case of mica, the laminary diffraction fringes produced by the striae often showed distinct asymmetry though to a much less extent than in the case of etched glass plates. The central fringe which is strongly coloured often showed distinctly different tints at its two edges and was occasionally even completely bifurcated in colour. It was thought that a closer examination of these phenomena, and of the nature of the laminar boundaries in mica, would be of interest. The results of the investigation are presented in this paper.

SECTION II.—MICRO-STRUCTURE OF THE STRIAE.

The striae appear to the naked eye as fine hair-like lines on the surface of the mica when the latter is examined in diffuse light. Under the microscope, however, an interesting structure is revealed and the striae appear resolved into minute echelons, or staircases, the number of steps in the echelon being often considerable. Under direct illumination and moderate powers, the boundaries of the successive steps of the echelon appear in the bright field as fine dark lines which when the highest powers of the

* R. W. Wood, "Physical Optics," 1914 Edition, page 250. The symmetry should be exact when the phase-difference on the two sides of the boundary is π or any multiple of π. In other intermediate cases, the pattern is asymmetrical, but in a relatively minor degree.

† See, for instance, the photograph by Wood published in his book, *loc. cit.*

microscope are used may either remain visible as such or appear still further resolved into two, three, or more fine dark lines. Under oblique illumination, we have the opposite effect, the entire field of the microscope being dark and the successive edges of the echelon appearing as bright lines. It should be noticed that the successive edges in a stria do not all appear equally dark when it is seen under direct illumination, nor do they all appear equally bright when seen under oblique illumination. It is thus clear that the successive steps of the echelon generally represent unequal changes of thickness in the mica. Further, the steps are also generally of unequal width, and the width of a step may even vary from point to point along the length of a stria. In some cases a single boundary may be seen split up along its length into two or even into three boundaries. The width of a stria is a variable magnitude. Striae have been observed of which the entire width does not exceed 1/400 mm. or even 1/600 mm., i.e., about five wave lengths of sodium light, and broad striae have been observed which extend over 1/20th or 1/15th part of a mm., that is, from one hundred to one hundred and fifty times this wave-length. The number of steps is in some cases fifteen or twenty but is usually found to be between six and ten, and may occasionally be as small as one, two, or three.

Taking the case of a typical stria whose micro-photograph is shown in fig. 1 in the Plate, we find it has seven steps, the total width of the steps being about 1/100 mm., i.e. about 20 wave-lengths.

The widths of the successive steps in this stria as actually measured with a micrometer were found to be as shown below:—

	1st Position.	2nd Position.	3rd Position.
1.	·0010 mm.	·0010 mm.	·0009 mm.
2.	·0025 ,,	·0022 ,,	·0023 ,,
3.	·0010 ,,	·0010 ,,	·0011 ,,
4.	·0015 ,,	·0017 ,,	·0018 ,,
5.	·0028 ,,	·0024 ,,	·0028 ,,
6.	·0022 ,,	·0026 ,,	·0010 ,,

With the Jamin interferometer it was found that the optical retardation produced by the mica on the two sides of this stria differed by just half a wave-length of sodium light. On the assumption that this retardation is equally distributed between the

successive steps, the difference of retardation at each step is of the order of 1/14th of the wave-length of the D lines. But the actual thicknesses are of different magnitudes as is evident from the appearance of the different boundaries as seen under the microscope, and some of the steps probably therefore represent optical retardations even less than 1/20 or 1/25th part of the wave-length of sodium light. It is remarkable that laminar boundaries representing such small optical retardations in a transparent plate are clearly seen as fine dark lines under direct illumination in the microscope. The visibility of the laminar edges is evidently closely connected with their power of diffracting light, and presents some interesting points of comparison with the question of the visibility of particles of much smaller dimensions than the wave-length of light in the microscope. It would seem worthwhile on a future occasion to examine the matter further and determine by direct observation the smallest thicknesses of the laminar boundaries which can be obtained in mica and which can be detected in the microscope under direct or indirect illumination.

The appearance of the laminar edges in etched plates of glass under the microscope forms a striking contrast with the phenomena observed in mica. They appear as blurred and ill-defined bands even under low powers, and become altogether indistinguishable when objectives of higher powers are used.

SECTION III.—SPECTRUM OF THE LAMINARY DIFFRACTION PATTERN.

As remarked in the introduction, the distribution of intensity and colours in the diffraction-patterns produced by the striae in mica often shows a distinctly noticeable asymmetry of which the explanation is evidently connected with the echelon-like nature of the boundary. A delicate method of observation which would render this asymmetry strikingly evident and susceptible of measurement became necessary and was ultimately found in the spectroscopic examination of the diffraction-pattern. When the laminar diffraction fringes are allowed to fall crosswise on the slit of a spectroscope, the spectrum as seen is found to be crossed by dark bands showing the colours obscured by interference in the different parts of the pattern. If the distribution of intensity in the laminar diffraction-pattern had been strictly symmetrical, we

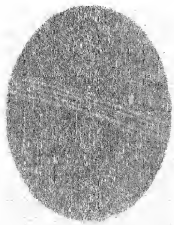


Fig. 1

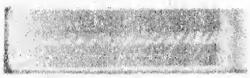


Fig 2

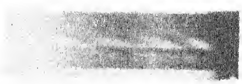


Fig. 3

Fig. 4

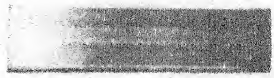


Fig. 5

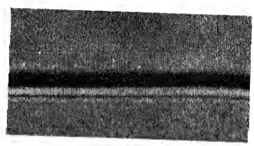


Fig. 6

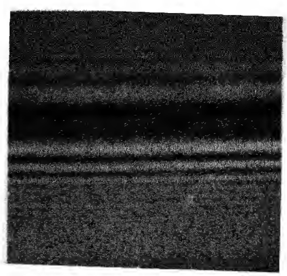


Fig. 7

Laminar Diffraction phenomena observed with Mica.

should evidently see in the spectrum a system of bands with a configuration symmetrical about the central line. Actually, however, we obtain the very curious appearance of bands running obliquely through the spectrum (Figs. 2, 3, 4 in Plate IV).* Fig. 2 which shows quite a large number of bands was obtained with a rather thick stria which showed merely a grey colour in the central fringe. Here every one of the bands is inclined, the inclination gradually diminishing as one proceeds towards the violet end of the spectrum. The actual shape of the individual bands running through the spectrum is an oblique *ſ* of which the tips are considerably fainter than the middle. The same characteristics are also shown by fig. 3 in which we find four oblique bands distributed practically equally throughout the spectrum, and the distribution of intensity above and below these bands in the spectrum is obviously quite different. The phenomena are still more clearly seen in fig. 4 which was obtained with a stria which gave a green-coloured central fringe. The obliquity of the two dark bands running through the spectrum and the asymmetric distribution of intensity above and below these bands is particularly noticeable in this figure. Fig. 5 represent sthe spectrum of the fringes due to a stria giving a half wave-length change in optical retardation. Here, though the band runs straight through the spectrum, a distinct obliquity in its position and an asymmetric distribution of intensity on the two sides of it were both noticed. The intensity was greater on one side of the fringe than on the other, and the fainter bands which accompany the central dark band on either side are much clearer and further from it on one side than on the other.

The relative optical retardation on the two sides of a stria should obviously be increased by tilting the mica, and it is interesting to watch the effect of this on the position and number of the bands in the spectrum of the laminary diffraction-pattern. The observations are best made with a stria which shows a small number of bands in the spectrum. It is found that as the mica is

* It is found that superposed upon these bands, there also appear in the spectrum a set of very fine, numerous and rather diffuse bands rnnning parallel to the slit of the spectroscope. These are the well-known bands due to the interference of the light transmitted through the mica with the light which has passed into the spectroscope after two or more internal reflections within the mica, and need no further remarks.

turned gradually from the normal to the oblique position with respect to the light incident on it, the bands shift towards the red, and the number of the bands visible in the spectrum also increases.

Phenomena which are roughly analogous to the above but not so beautifully clear and regular are obtained when the spectrum of the laminar diffraction of a very thinly etched glass plate is observed. The appearance of asymmetry in this case is considerably more exaggerated, the fringes on one side of the pattern being numerous and showing well-marked contrasts, and the fringes on the other side being few in number and hazy in outline.

SECTION IV.—EFFECTS OBSERVED CLOSE TO THE STRIAE.

The phenomena described in the preceding section are those observed at comparatively large distances (of the order of a meter) from the mica. It is of interest to examine the effects observed near the sheet of mica, say within a few centimeters of it, as we may naturally expect that the asymmetry of the pattern due to the finite width of the striae would be much more marked in their immediate neighbourhood. For the purpose of this study, it is found convenient to use striae which run more or less straight. The source of light is a slit placed at a sufficient distance from the mica and parallel to the direction of the striae, and illuminated by sunlight or the light of an electric arc. The diffraction-fringes formed in the rear of the striae may be observed through a microscope. For spectroscopic analysis of the pattern, the eyepiece of the microscope may be removed and the image of the diffraction-pattern formed by the objective may be allowed to fall on the slit of a direct-vision spectroscope or of a constant-deviation wave-length spectrometer. By racking out the objective of the microscope, the complete succession of phenomena commencing from the plane of the mica itself right up to any desired distance from it may be observed in quick succession, and a vivid idea obtained of the whole series of effects. The observations made in this manner, especially with the aid of the spectroscope as described above, are extremely useful in getting at a clear understanding of the whole case.

The phenomena noticed naturally depend a good deal on the particular stria under study, especially on the number, width, and height of the steps in the staircase and the total relative optical

retardation on the two sides of it. Nevertheless, certain general features are observed which are common to most of the cases studied. When the focal plane of the objective coincides as nearly as possible with the mica, the structure of the stria is clearly seen, the successive edges of the echelon appearing as fine dark lines. Even at this position, however, a number of very fine (practically equidistant) fringes may be seen bordering these edges, these being more marked on one side of each edge than on the other. The simplest case is that in which the echelon consists of but a single step. Occasionally striae may be obtained consisting of but a single edge not resolvable by a 1/8th inch Reichert objective. Even in such cases, however, the laminar diffraction-pattern at close quarters often shows a distinct and sometimes quite marked asymmetry, the fringes on one side being clearer, brighter and more numerous than on the other, and the central fringe appearing much darker than it is at a distance from the mica. These features persist till the focal plane is drawn away from the mica to a distance of a centimeter or two. Apart from these special characters, however, the pattern due to an echelon of a single step is very similar to what we should expect on the elementary theory in the case of a perfectly abrupt laminar boundary.

With striae consisting of two, three or more steps in the staircase, the effects are naturally more complex than in the case of a single step. As the focal plane is drawn away from the mica, the fringes bordering the successive edges rapidly broaden out and become superposed on each other. Soon, all trace of the structure of the stria is lost owing to this superposition, and the field of view may be then seen clearly differentiated into three parts the effects observed in which may be separately considered.

First, the central part of the field which is much darker than the rest of the field. The width of this part of the field differs for different striae, being large for the broad striae, and small for the narrow ones. This central region is seen filled with a succession of faint but markedly coloured fringes which are at first narrow and numerous, but widen out and become fewer in number as the focal plane is drawn away from the mica, till finally only one or two coloured bands remain in the middle of the field.

Secondly, the part of the field lying on the thicker side of the stria. This is the brightest part of the field, and contains numer-

ous well-marked fringes which at first are nearly equally spaced and extend to a considerable distance from the central part of the field. As the focal plane is drawn away from the mica, these fringes become broader, less bright, and their spacing becomes more unequal. Fewer fringes are also then visible.

Thirdly, the part of the field on the thinner side. This usually contains only a few faint and hazy diffraction fringes. The contrast between the second and third parts of the field is very marked but decreases gradually as we recede from the mica.

Corresponding to the foregoing changes in the microscopic appearance of the fringes, the spectrum of the diffraction-pattern also alters. In the central part of the field, a number of narrow *horizontal* bands is at first seen in the spectrum. As the objective recedes from the mica, these bands widen out, those on the thicker side of the mica slide away towards the red end of the spectrum, those on the thinner side slide away towards the blue end, and the remaining bands gradually assume an oblique position and tend to set themselves in the spectrum less inclined to the slit of the spectroscope as we recede from the mica. When the focal plane is within a centimeter or two of the mica, a large number of very dark and bright fringes may be seen in the region of the spectrum on the thicker side of the stria, and relatively few and more hazy fringes on the thinner side. These fringes in the spectrum gradually widen out and become less numerous and more hazy in outline as we recede from the mica.

Figs. 6 and 7 in the Plate illustrate the preceding remarks and represent microphotographs (in the light of the electric arc) of the diffraction-effects observed close to the striae, fig. 6 having been secured with the focal plane nearer the mica, and fig. 7 with the focal plane somewhat further away. As the light used was not monochromatic, the photographs do not convey an adequate idea of the very large number of fringes visible when the pattern is analysed by the spectroscope.

The diffraction-effects due to the laminar boundaries in etched glass plates observed in their neighbourhood through a microscope are not so striking as those observed with mica, the number of fringes visible, specially in the spectral analysis of the pattern, being much smaller than with mica. Apart from this, however, the effects observed in the two cases are broadly analogous,

the asymmetry being considerably more exaggerated in the case of etched glass plates.

Section V.—Intensity, Colour and Polarization of the Large-Angle Diffraction.

On account of the extremely fine structure of the laminar edges forming the striae in mica, they possess the property of scattering or diffracting light in directions making large angles (up to 180°) with the light transmitted through or reflected from the plane surface of the mica. The feature of this large-angle diffraction which is most noteworthy is that the striae as seen by the light diffracted by them appear very *much more intensely* luminous on the *retarded* side of the wave-front and relatively quite feebly luminous on the other side. Observation also shows that when white unpolarized light is incident on the striae, the light scattered by them is often strongly coloured and also polarized. The investigation of these effects is a matter of some complexity, as they are found to depend on a number of factors, (*a*) the fine structure of the laminar edge and the relative optical retardation on the two sides of it, (*b*) the angle of incidence of light, and (*c*) the angle of diffraction. Further, though the problem of the large-angle diffraction of light by the edge of a semi-infinite *perfectly reflecting* screen has been investigated by Sommerfeld and shown to involve polarization effects, the corresponding problem of the large-angle diffraction by the edge of a thin lamina of transparent solid yet remains unsolved, and no theoretical guidance for research in this direction is therefore available. It is hoped to investigate these effects in detail experimentally when a suitable opportunity occurs.

Section VI.—Mathematical Theory.

The observations considered in the preceding sections which require explanation are the following: (*a*) the appearance and visibility of the laminar boundaries in the microscope when focussed upon them in direct illumination; (*b*) the diffraction phenomena observed close to the striae, and (*c*) the effects observed at a distance, especially the obliquity of the bands appearing in the spectrum of the diffraction-pattern. The investigation of (*a*) is a question relating to the theory of microscopic vision

which the author hopes to be able to deal with in a later paper. The investigation of (*b*) and (*c*) strictly speaking requires an exact knowledge of the structure of the particular stria under observation. As our present purpose is, however, merely to get a general idea of the explanation of these effects, it is sufficient if the stria is assumed to be an approximately wedge-shaped boundary separating regions of the mica having slightly different thicknesses. Such a wedge-shaped boundary would diffract light in roughly (though not by any means in exactly) the same way as the equivalent staircase structure.

Considering now the effects in the neighbourhood of the

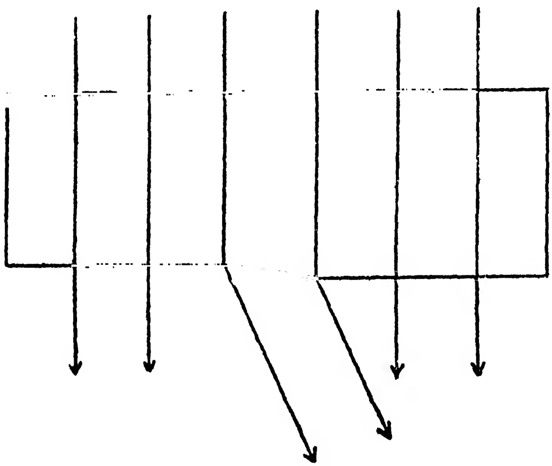


FIG. 8.

stria, fig. 8 showing the geometrical path of the rays gives us at once a general idea of the phenomena to be expected. The central relatively dark part of the field, the increased brightness and the numerous interference fringes seen in the region on the thicker side of the mica, and the relatively few and hazy diffraction fringes seen on the thinner side of the mica are all exactly what we should expect on the principles of the wave-theory. The following is the detailed mathematical treatment :—

The form of the wave front on emergence from the plate shown in fig. 8 would be that shown by the upper line in fig. 9. The effect at any point O in the field may be readily found in terms of Fresnel's

integrals. In fig. 9, r_0 is the distance of the pole from the point of observation and x_1, x_2 the distances of the two edges of the stria from the pole. ρ is the relative retardation of the wave front on the two sides of the stria.

We may write $\sigma = \dfrac{\rho}{x_1 - x_2}$

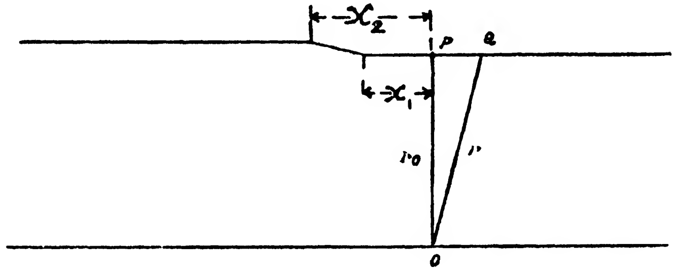


FIG. 9.

We then get as the expression for the amplitude omitting constants

$$\int_{-x_1}^{\infty} \cos 2\pi \left(\frac{t}{\tau} - \frac{r}{\lambda} + \frac{1}{8}\right) dx + \int_{-\infty}^{-x_2} \cos 2\pi \left(\frac{t}{\tau} - \frac{r}{\lambda} - \frac{\rho}{\lambda} + \frac{1}{8}\right) dx$$

$$+ \int_{-x_2}^{-x_1} \cos 2\pi \left(\frac{t}{\tau} - \frac{r}{\lambda} - \frac{\sigma(x_1 - x)}{\lambda} + \frac{1}{8}\right) dx$$

Putting

$$\frac{t}{\tau} - \frac{r_0}{\lambda} + \frac{1}{8} = A$$

$$\frac{\rho}{\lambda} = \theta_1$$

$$\frac{\sigma x_1}{\lambda} - \frac{r_0 \sigma^2}{2\lambda} = \theta_2$$

$$x - r_0 \sigma = z$$

$$- z_1 = - x_1 + r_0 \sigma$$

$$- z_2 = - x_2 + r_0 \sigma$$

the expression for the amplitude becomes

$$\int_{-x_1}^{\infty} \cos 2\pi \left(A - \frac{x^2}{2r_0\lambda}\right)dx + \int_{-\infty}^{-x_2} \cos 2\pi \left(A - \theta_1 - \frac{x^2}{2r_0\lambda}\right)dx$$

$$+\int_{-z_2}^{-z_1} \cos 2\pi \left(A - \theta_2 - \frac{z^2}{2r_0\lambda}\right)dx$$

which may be transformed into the usual form of Fresnel's integrals by making

$$v_1 = x_1\sqrt{\frac{2}{r_0\lambda}} \qquad v_3 = z_1\sqrt{\frac{2}{r_0\lambda}}$$

$$v_2 = x_2\sqrt{\frac{2}{r_0\lambda}} \qquad v_4 = z_2\sqrt{\frac{2}{r_0\lambda}}$$

We get the expression for amplitude to be

$$\cos 2\pi A \int_{-v_1}^{\infty} \cos \tfrac{1}{2} \pi v^2 dv + \sin 2\pi A \int_{-v_1}^{\infty} \sin \tfrac{1}{2} \pi v^2 dv$$

$$+\cos 2\pi (A - \theta_1) \int_{-\infty}^{-v_2} \cos \tfrac{1}{2} \pi v^2 dv + \sin 2\pi (A - \theta_1) \int_{-\infty}^{-v_2} \sin \tfrac{1}{2} \pi v^2 dv$$

$$+\cos 2\pi (A - \theta_2) \int_{-v_4}^{-v_3} \cos \tfrac{1}{2} \pi v^2 dv + \sin (A - \theta_2) \int_{-v_4}^{-v_3} \sin \tfrac{1}{2} \pi v^2 dv$$

putting

$$a_1 = \int_{-v_1}^{\infty} \cos \tfrac{1}{2} \pi v^2 dv, \quad a_2 = \int_{-v_1}^{\infty} \sin \tfrac{1}{2} \pi v^2 dv$$

$$b_1 = \int_{-\infty}^{-v_2} \cos \tfrac{1}{2} \pi v^2 dv, \quad b_2 = \int_{-\infty}^{-v_2} \sin \tfrac{1}{2} \pi v^2 dv$$

$$c_1 = \int_{-v_4}^{-v_3} \cos \tfrac{1}{2} \pi v^2 dv, \quad c_2 = \int_{-v_4}^{-v_3} \sin \tfrac{1}{2} \pi v^2 dv$$

We get the following expression for the intensity

$$J^2 = (a_1 + b_1 \cos 2\pi\theta_1 + c_1 \cos 2\pi\theta_2 - b_2 \sin 2\pi\theta_1 - c_2 \sin 2\pi\theta_2)^2$$

$$+ \ (a_2 + b_1 \sin 2\pi\theta_1 + c_1 \sin 2\pi\theta_2 + b_2 \cos 2\pi\theta_1 + c_2 \cos 2\pi\theta_2\ )^2$$

In the particular case of a stria whose spectrum of laminary diffraction is illustrated in fig. 4 in the plate,

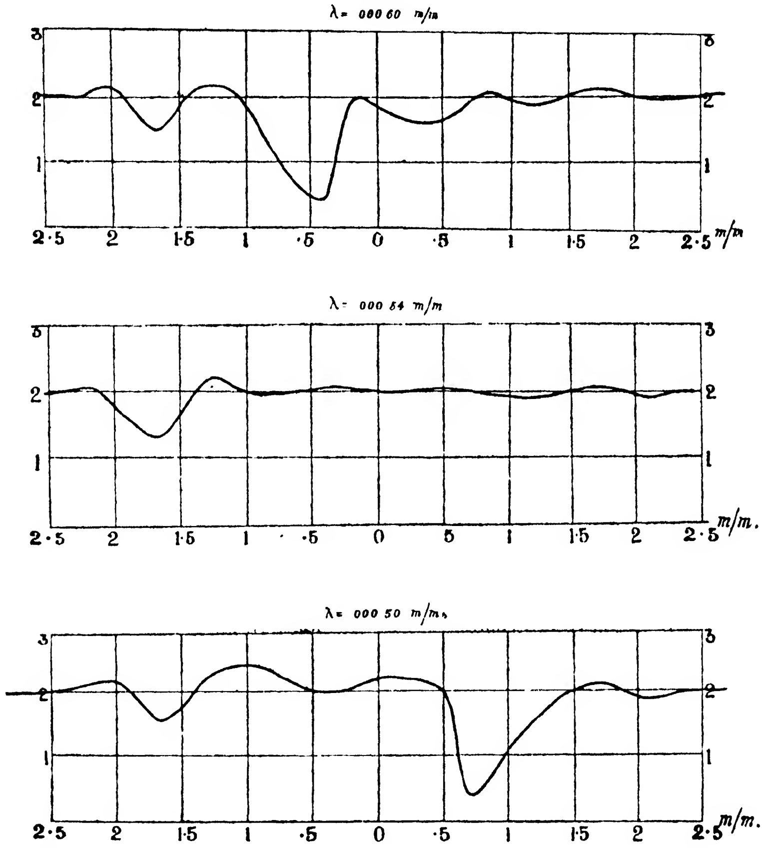
$$\rho = {\cdot}00054 \text{ m/m}$$
$$x_1 - x_2 = {\cdot}025 \text{ m/m}$$
$$r_0 = 1 \text{ meter}$$

the illumination curves for three wave lengths

$$\lambda = {\cdot}00050 \text{ m/m}$$
$$\lambda = 00054 \text{ m/m}$$
$$\lambda = {\cdot}00060 \text{ m/m}$$

have been calculated and plotted (fig. 10).

Fig 10



They clearly show that

(1) for wave length $\lambda = {\cdot}00050$ m/m the central dark fringe has somewhat shifted to one side;

(2) for wave length $\lambda = {}^{\cdot}00054$ m/m the central dark fring has vanished;

(3) for wave length $\lambda = {}^{\cdot}00060$ m/m the central dark fringe has shifted to the opposite side.

This is precisely the effect which appears in fig. 4 in the plate on the two sides of the transmission region in the spectrum.

SECTION VII.—SYNOPSIS.

In a previous paper (published in the Proc. Roy. Soc. for November 1919), the author has described and explained the colours shown by the striae or laminar boundaries in mica when the same is examined by the Foucault test. The present paper describes some observations on the microstructure of these laminar boundaries and of the various diffraction effects produced by them. The following are the principal results obtained.

(*a*) The striæ appear resolved in the microscope into minute echelons or staircase structures, the number of the steps varying from one to ten or fifteen for different striae. The optical retardation due to any particular edge in the echelon is generally quite a small fraction of a wave length. The edges are nevertheless clearly visible in the microscope as very sharp dark lines.

(*b*) In consequence of the structure of the striae above mentioned, the laminar diffraction pattern observed even at a considerable distance from the mica shows distinct evidence of asymmetry in the distribution of intensity and colour of the fringes.

A very delicate method of exhibiting this asymmetry is furnished by spectroscopic analysis of the laminar diffraction pattern, the dark bands due to interference running obliquely through the spectrum and being much more clearly marked on one side of the pattern than on the other.

(*c*) In the immediate neighbourhood of the striæ the diffraction-phenomena as observed through a microscope are more complicated, the asymmetry being very marked, and a very large number of fringes may be observed.

(*d*) The striæ scatter light through large angles, the light thus diffracted showing a marked asymmetry in its intensity on the two sides of the direction of the incident light and also exhibiting both colour and polarisation.

The investigation described in this paper was carried out in the Palit Laboratory of Physics, and the author's best thanks are due to Prof. C. V. Raman for his suggestions and unfailing interest in the work during its progress.

V. On the Forced Oscillations of Strings under Damping proportional to the Square of the Velocity.

By Rajendra Nath Ghosh, M.Sc., Research Scholar of the Association.

(Plate V.)

CONTENTS.

SECTION I.—INTRODUCTION.

In a recent paper published in the *Physical Review* * J. Parker Van Zandt has given a discussion and comparison with experiment of the theory of *free* oscillations of bodies subject to resisting forces proportional to the *square* of the velocity. He has also given a very full bibliography of the literature on oscillations subject to this and other special laws of damping. In the course of some quantitative work on the forced oscillations of strings under different types of excitation recently undertaken by the present author, results have been obtained which show that the frictional forces acting on a stretched string vibrating in air are not propor-

* *Physical Review*, Nov. 1917, page 415.

tional to the velocity of each point on the string as is generally assumed,* but increase much more rapidly than in proportion to the velocity. It accordingly appeared to be of interest to investigate the theory of forced oscillations under damping proportional to the square of the velocity, and to compare the results with those found in experiments on stretched strings vibrating under different types of excitation. This has been done, and the present paper describes the outcome of the investigation. The experimental work has been carried out with the aid of the apparatus for the study of vibrations of strings developed by Prof. C. V. Raman and described in a recent publication.† This apparatus is specially suitable for the present investigation and has indeed made it possible.

SECTION II.—LAW OF DAMPING OF FREE VIBRATIONS OF A STRETCHED STRING.

Observations have been made by the author of the rate of decay of the free oscillations of a stretched cord (of twisted cotton thread) vibrating in air. This was investigated by a photographic method. The middle point of a stretched string was pulled aside and then released. The damped oscillation of the point was photographed on a moving plate, and measurements of the amplitudes were made by means of a cross-slide micrometer. The following table shows a typical set of results. In column I we have the successive amplitudes, in column II their successive differences D, in column III the mean M of two successive amplitudes.

---

* Lord Rayleigh, "Theory of Sound," Vol. I, Arts. 131-134.

† C. V. Raman, "An Experimental Method for the Production of Vibrations," *Physical Review*, Nov. 1919.

TABLE I.

Length of String 100 *cms.*

No.	Successive amplitudes.	D	M	D/M²	D/M
	cms.				
1	2·0344 / 1·6785	·3559	1·8564	·13	·19
2	1·4102	·2683	1·5443	·13	·17
3	1·2115	·1982	1·3108	·11	·15
4	1·0363	·1752	1·1239	·13	·15
5	0·9069	·1294	0·9716	·14	·13
6	0·8147	·0922	0·8608	·12	·11
7	0·7050	·1097	0·7568	·13	·14
8	0·6424	·0626	0·6737	·14	·09

In the fourth column it will be observed that D/M² is almost constant whereas D/M shows a variation much greater than the experimental errors. Now we know that when the damping is proportional to the first power of the velocity, the ratio of two successive amplitudes is constant, from which it can be easily shown that the difference of two successive amplitudes divided by their mean must also be a constant. But in the present case it is seen that the above ratio namely D/M is not constant, while D/M² is constant. Hence the law of frictional resistance is not that of the first power of the velocity. We shall presently show that the result D/M² is a constant expresses the fact that the frictional forces are proportional to the square of the velocity.

The equation of motion of a particle on the stretched string vibrating freely and resisted by forces proportional to the square of the velocity is given by

$$\ddot{U} = -n^2 U \pm k\dot{U}^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

where U is the displacement and the constants have their usual meanings. For a first approximation, neglect the term $\pm k\dot{U}^2$, and we have

$$U = A \cos nt \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

By the method of successive approximation we get the value of U when resisting forces are taken into consideration.

$$U = A \cos nt + \tfrac{1}{2} A^2 k + \tfrac{1}{6} A^2 k \cos 2\,nt \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let $U = A_1$ and $\dfrac{dU}{dt} = o$ when $t = o$

Therefore we get $A = A_1 - \frac{2}{3} k A_1^2$ and U is then given by

$$U = (A_1 - \tfrac{2}{3} k A_1^2) \cos nt + \tfrac{1}{2} A_1^2 k + \tfrac{1}{6} A_1^2 k \cos 2nt \quad . \quad . \quad . \quad (4)$$

Equation (4) is only true for half an oscillation. At $t = \pi/n$ the displacement U is given by

$$U = -(A_1 - \tfrac{4}{3} A_1^2 k)$$

Hence the amplitude of swing is diminished by $\frac{4}{3} A_1^2 k = D$ (approx.). Hence we arrive at once at the result that

$$D/M^2 = \tfrac{4}{3} k \qquad (5)$$

From the experimental results we have seen that D/M^2 is almost a constant. Hence we infer that the frictional forces for the large amplitudes of vibration employed are proportional to the square of the velocity.

From other results obtained by the author it would seem that for very small amplitudes of vibration on the other hand, the frictional forces do not increase quite so rapidly as in proportion to the square of the velocity, and are more nearly proportional to the first power.

SECTION III.—EXPERIMENTAL STUDY OF FORCED VIBRATIONS OF STRINGS.

Description of apparatus.

For an experimental study of the forced oscillations of stretched strings, an electric motor-vibrator was used. A detailed account of this apparatus as originally devised by Professor J. A. Fleming, and subsequently modified and improved by Professor C. V. Raman, has been published in the *Physical Review*.* In Plate V, Fig. 1 shows the general features of the apparatus. A circular wheel is fixed to the axle of a small motor. To this wheel a brass disk carrying a slot and movable pin is fixed. The pin

* November, 1919.

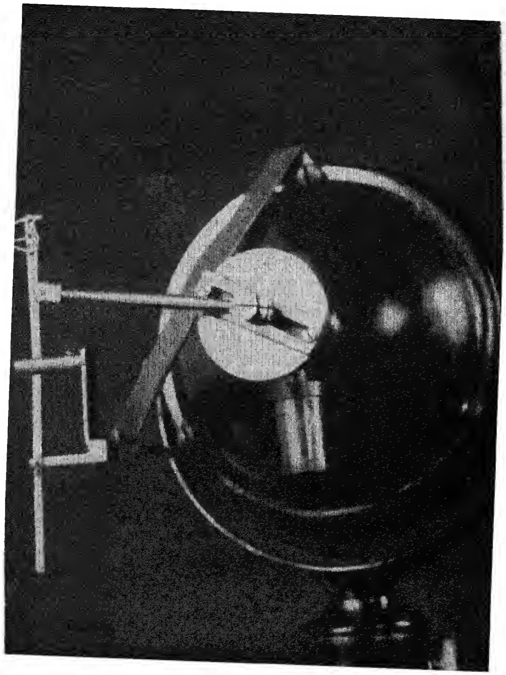


Fig. 1

Fig. 2

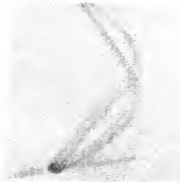


Fig. 3

Forced Oscillations of Strings.

actuates a crank shaft and oscillating lever moving between guides. The oscillating lever performs what is approximately a simple harmonic motion, and excites the vibration of the string in the manner of Melde's experiment. The frequency of this excitation depends upon the rate of revolution of the motor; and the amplitude of motion of the lever-arm can be governed to a nicety by altering the position of the pin on the motor. When the pin is exactly at the centre, the amplitude of motion of the lever is zero, and as the pin is moved away from the centre, the amplitude increases. This adjustability of the amplitude enhances the utility of the apparatus. The frequency of the vibrator, i.e. of the lever arm, was kept constant by a sliding rheostat and the constancy was observed through a stroboscopic tuning fork. In all the experiments, the frequency was generally 60 per second. The tension was adjusted by putting weights on a pan and clamping the end of the string near it, so that the pan could not oscillate. The motor-vibrator permits the use of long strings, considerable tensions, and gives a perfectly constant amplitude, advantages which are not possible when a tuning fork is used as the exciter.

The amplitude of vibration of the string depends on (1) its tension and (2) the magnitude of the obligatory motion imposed on it at the end. Keeping (2) constant, (1) may be varied and the resonance curve of the string may be traced. This may be done for a series of values of (2).

Transverse Excitation.

The case first studied in the manner described above was the transverse type of Melde's experiment in which the maintained oscillation has the same period as the obligatory motion impressed at one end. This has been discussed mathematically by the late Lord Rayleigh * on the assumption that the frictional resistances are proportional to the first power of the velocity. Corresponding to an obligatory motion $Y=\gamma \cos pt$ at a nodal point, the amplitude of vibration of the string at any point is approximately given by

$$A_x=\gamma \left\{ \frac{\sin^2 \frac{2\pi}{\lambda} x + \frac{k^2 x^2}{4a^4} \cos^2 \frac{2\pi}{\lambda} x}{k^2 b^2/4a^2} \right\}^{\frac{1}{2}}.$$

* Theory of Sound, Vol I, Art 131

so that we should find the amplitude of vibration to be proportional to the obligatory motion γ. Actually, however, on determining the amplitude of vibration of the string for the different values of

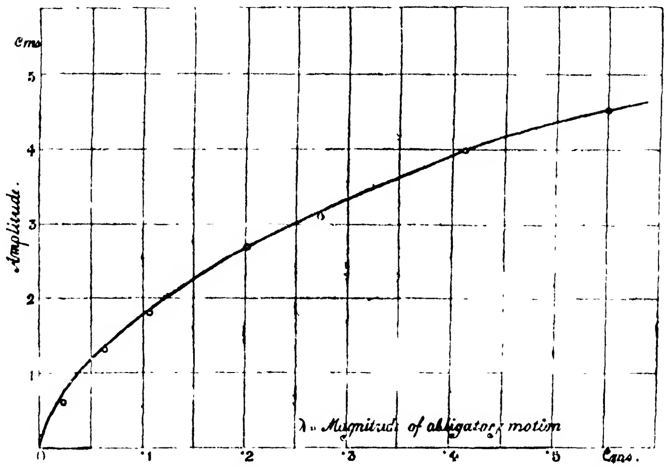


FIG. 4.—Relation between Amplitude and Obligatory Motion.

γ, the amplitude increases much less rapidly than in proportion to γ. The experimental results are shown in fig. 4 in which the

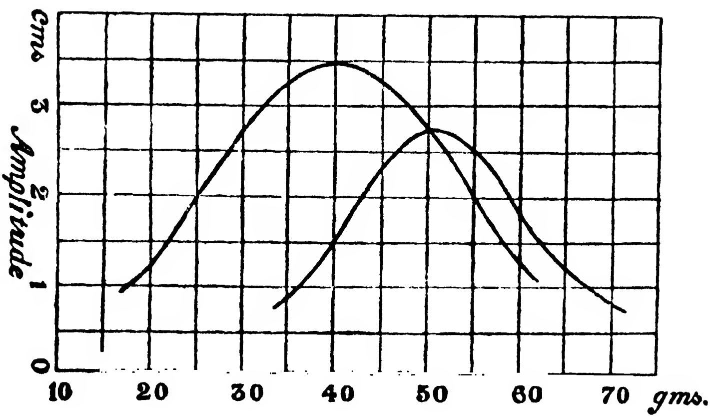


FIG. 5.—Resonance Curves for two different values of γ.

ordinates represent the maximum amplitudes of vibration obtainable with the respective extents of the obligatory motion shown as abscissae. The form of the graph is approximately a parabola.

This result indicates that the frictional resistance is not proportional to the first power of the velocity, but, as I shall show later on, is more nearly in proportion to the square of the velocity.

Another important fact noticed was that the tension for which the maximum amplitude is obtained decreases with increasing values of γ. This point was studied by adjusting the tension of the string, clamping its free end, and then starting the motor vibrator. The experimental results are shown in fig. (5) where the amplitudes of vibration of the string are plotted against the tension for two different values of γ. The shift of the maximum amplitude and of the whole curve when the amplitude of vibration is large, is evident from a comparison of the two curves. The theoretical interpretation of this result will be discussed later in the course of the paper.

Longitudinal Excitation.

The next case studied was the longitudinal type of Melde's experiment in which the obligatory motion imposed at one end varies the tension of the string. In this case, as has been shown by C. V. Raman,* maintenance is possible when the frequencies of the obligatory motion and of the oscillation maintained stand in any of the ratios 2 : n where n is an integer. In the present paper attention will be confined to the first case in which the frequency ratio is 2 : 1. The theory of this case was first given by the late Lord Rayleigh, and later modified and extended by Prof. Raman in the papers cited so as to give results more in accordance with experiment, the assumption being made that the frictional force is proportional to the first power of the velocity. The experimental observations of Prof. Raman were made using a tuning fork as exciter, and as the reaction of the string on the fork materially influences the vibration of the fork and alters its amplitude, a difficulty arises in interpreting the results observed. With the electric motor-vibrator, on the other hand, the amplitude of the obligatory motion is invariable, and its frequency can be kept constant by the use of a rheostat so that the reaction of the string can exercise no effect. A stricter comparison of the

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* Bulletin No. 6 of the Association, also *Phil. Mag.*, October 1912, and *Physical Review*, December 1912, July 1914 and November 1919; see also Jones and Phelps, *Physical Review*, November 1917.

experimental results with theory is therefore possible. The experimental observations may be considered under two heads, (*a*) dependence of amplitude of vibration upon the tension and the magnitude of its periodic variation, and (*b*) dependence of the phase of vibration on the same variables.

(*a*): In order to study the relation between the amplitude of vibration of the string and the tension, the latter was made very high at the beginning, and then it was continuously diminished. With the change in tension, the speed of the motor and of the variation of tension produced by it tend to change to a slight extent, but they were kept constant as explained above

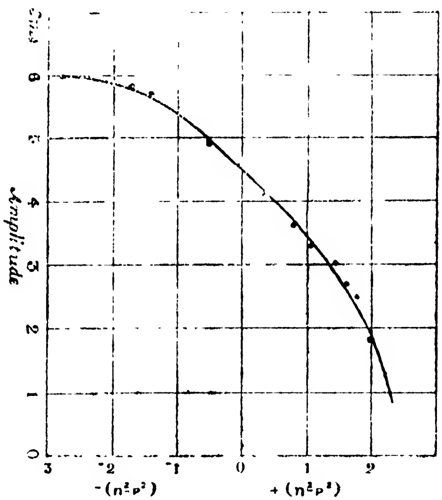

Fig. 6.—Resonance Curve under Variable Spring of Double Frequency.

with the aid of a sliding rheostat and a stroboscopic fork. It was found that at a high tension (much greater than the value for which the natural frequency of vibration of the string is half that of the vibrator) the amplitude of vibration is small, but has always a definite value for a definite tension. As the tension is diminished, the amplitude of vibration increases, and continues to increase till a stage is reached when the tension is much lower than the theoretical value for which the natural frequency is half that of the vibrator; the motion of the string then suddenly collapses giving place to other types of oscillation. At this instant the amplitude is a maximum. When the tension is such that

the natural frequency is half that of the vibrator, the amplitude is much less than the maximum value. The amplitude is not the same when the tension is greater or less than the theoretical value for half frequency by a certain amount. The whole sequence of events is shown in fig 6. The graph shown there has been drawn from the experimental data. The continuous increase of the amplitude with the diminution of tension, and the consequent asymmetric character of the curve are both clearly shown. The largest value of the amplitude obtained at the collapse point is shown by one extremity of the curve.

As the amplitude of the vibrator and consequently also the magnitude of the periodic variation of tension are increased, the maximum possible amplitude of vibration of the string also increases. The experimental curve simply shifts up through a certain distance. If on the other hand we diminish the magnitude of the imposed periodic variation of tension, the amplitude maintained also diminishes, and when the imposed variation falls below a certain very small limit no maintenance is obtained.

(*b*): The phase relation between the exciting force and the vibration of the string was easily studied by attaching a bright bead to the string near the vibrator. Each point of the string has two motions at right angles to each other, the longitudinal motion being due to that of the vibrator, and the other the transverse motion of the string itself. Hence the Lissajous figure traced by the bead gives the phase relation required. Using arc-light to illuminate the bead, the Lissajous figures were photographed at different amplitudes of vibration of the string. Fig. 2 in Plate V shows the figure when the amplitude was small corresponding to a high tension. The parabolic figure seen (convex towards the vibrator) at once indicates the frequency and phase relations between the longitudinal and transverse components of the motion, in other words between the maintaining force and the maintained vibration. Fig. 3 in the Plate shows the Lissajous figure when the amplitude of vibration of the string was large corresponding to a low tension. The change in phase which has occurred in consequence of decrease of tension is evident. The rapidity with which the change occurs depends upon the magnitude of the imposed variation of tension. When this is large, we find that the phase relation alters slowly till the form of the

Lissajous figure which is originally a parabola convex to the vibrator reaches the limiting form of an 8 curve. At this stage the vibration collapses. On the other hand when the magnitude of the periodic variation of tension is small, the phase changes rapidly with decrease of tension, the initial and final forms of the Lissajous figures are the parabola and the 8 curve respectively in the same way as in the case previously mentioned.

SECTION IV.—THEORY OF FORCED OSCILLATION UNDER TRANSVERSE EXCITATION.

In this section we shall deal with (1) the forced vibration of a simple system of one degree of freedom, (2) the forced vibration of a stretched string at a point of which an obligatory motion is imposed, the damping in either case being assumed to be proportional to the square of the velocity.

Case (1). The equation of motion of a system of one degree of freedom, subject to frictional forces proportional to the square of the velocity, and acted on by a periodic force a sin pt, is given by

$$\ddot{U} \pm k\,\dot{U}^2 + n^2 U = a \sin pt \qquad (6)$$

the alternative signs being necessary as otherwise the frictional forces do not change sign with the velocity.

The solution of this equation will be of the form

$$U = A_1 \sin(pt+\epsilon_1) + A_3 \sin(3pt+\epsilon_3) + A_5 \sin(5pt+\epsilon_5) + \ldots \qquad (7)$$

Substituting (7) in (6) we get the resistance term as

$$\pm k\,\dot{U}^2 = \pm kp^2 \{A_1^2 \cos^2(pt+\epsilon_1) + 6\,A_1 A_3 \cos(pt+\epsilon_1)\cos(3pt+\epsilon_3) + 9\,A_3^2 \cos^2(3pt+\epsilon_3) + \ldots\} \qquad (8)$$

Retaining terms proportional to $\pm A_1^2$ and applying the Fourier analysis to get the periodic components of the frictional force, we get

$$\pm k\dot{U}^2 = \frac{2}{\pi} kp^2 A_1^2 \left\{ \frac{4}{3} \cos(pt+\epsilon_1) - \frac{4}{15} \cos(3pt+3\epsilon_1) - \frac{4}{105} \cos(5pt+5\epsilon_1) \right\} \qquad (9)$$

The series in (9) is a rapidly converging one. Substituting (9) in (6), expanding and equating the coefficients of sin pt and cos pt to zero, we get the following relations :—

$$A_1^2 = \frac{(\Delta^4 + 4\,a^2\,k_1^2 p^4)^{\frac{1}{2}} - \Delta^2}{2\,k_1^2\,p^4} \tag{10}$$

$$\tan E_1 = -\left\{\frac{(\Delta^4 + 4\,a^2\,k_1^2\,p^4)^{\frac{1}{2}} - \Delta^2}{2\,\Delta^2}\right\} \tag{11}$$

where $$\Delta = n^2 - p^2 \text{ and } k_1 = \frac{8k}{3\pi}.$$

The energy of the forced oscillation is a maximum when $\Delta = 0$, i.e. when $n = p$. The amplitude is then given by

$$A_1 = \frac{1}{p}\left(\frac{a}{k_1}\right)^{\frac{1}{2}} \tag{12}$$

These expressions show that the amplitude of the maintained oscillation at the peak of the resonance curve is proportional to the square root of the magnitude of the impressed force instead of the first power as in the ordinary case. Another special feature of the forced oscillation for this law of resistance is that the phase of the maintained motion, except at the peak of the resonance curve, varies with the magnitude of the impressed force, as can be seen from equation (11).

Defining *sharpness of resonance* as the square root of the quotient of the curvature at the peak of the resonance curve*, divided by the energy at the same position, we get

$$\textit{Sharpness of resonance} = \frac{n}{(k_1 a)^{\frac{1}{2}}}$$

So that we find that the sharpness of resonance is directly proportional to the frequency of oscillation of the system, and inversely proportional to the square root of the product of the impressed force and the damping coefficient, contrary to the case of ordinary resonance in which the sharpness of resonance defined in the same way is given by the equation †

$$\textit{Sharpness of resonance} = \frac{n}{k_1}$$

We have seen that the resonance is maximum when $n = p$,

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* The energy should be graphed as a function of the mistuning $\left(\frac{n}{p} - \frac{p}{n}\right)$.

† E. H. Barton, *Phil. Mag.*, July 1913.

but the amplitude of maintained motion is maximum when the frequency of the impressed force is equal to

$$p_1=(n^2-k_1a)^{\frac{1}{2}} \text{ (approx.)} \tag{13}$$

and is therefore less than the natural undamped frequency of the system. The difference of the squares of the frequencies natural to the system and that for which the amplitude is a maximum, is approximately proportional to the product of the damping coefficient and the impressed force, whereas in the ordinary case, the difference of the squares of the above frequencies is proportional to the square of the damping coefficient.

The maximum amplitude is given by

$$A_1=\left\{\frac{a}{k_1\,(n^2-k_1a)}\right\}^{\frac{1}{2}} \text{ approximately} \tag{14}$$

which shows that the maximum amplitude is proportional to the square root of the impressed force.

When the amplitude of vibration is large and the restoring force cannot be taken as strictly proportional to the displacement, we may modify the equation of motion, and write

$$\ddot{U}\pm k\dot{U}^2+(n^2+\beta U^2)\,U=a\sin pt \tag{15}$$

The displacement will be given by a series of the form

$$U=A_1'\sin(pt+\epsilon_1')+A_3'\sin(3pt+\epsilon_3')+A_5'\sin(5pt+\epsilon_5')$$

Applying the same method of analysis and proceeding as before we get equations similar to (10) and (11) etc., the only change necessary being made by writing Δ_1 for Δ in all the equations, where

$$\Delta_1=n^2-p^2+\tfrac{3}{4}\beta A_1'^2$$

The effect of a finite amplitude is therefore equivalent to an increase in the natural frequency of oscillation of the system. Hence the maxima of all the resonance curves will not be found when $n=p$, but for large amplitudes, they will shift towards a lower natural frequency of vibration.

Case (2). Now we shall pass on to the case of forced oscillation of a stretched string whose every elementary portion is resisted by forces proportional to the square of the velocity. The equation of motion of the string is given by

$$\frac{d^2y}{dt^2}\pm k\left(\frac{dy}{dt}\right)^2=a^2\frac{d^2y}{dx^2} \tag{16}$$

the alternative signs being necessary as explained before. The general solution of (16) is

$$y=u_1 \sin pt+v_1 \cos pt+u_3 \sin 3pt+v_3 \cos 3pt+\text{etc.} \qquad (17)$$

where the u's and v's are functions of x only.

Substituting (17) in (16) we get the frictional force of the form

$$\pm kp^2 \{u_1^2 \cos^2 pt+v_1^2 \sin^2 pt+ \ldots .\}, \text{ (approx.)}$$

Retaining terms proportional to u_1^2 and v_1^2 and analysing in Fourier series, we get

$$\pm kp^2 \{u_1^2 \cos^2 pt+v_1^2 \sin^2 pt\}$$

$$=-\alpha p^2 \{u^2 \cos pt+v^2 \sin pt+\text{terms of higher frequency}\} \quad (18)$$

where $\alpha=\dfrac{8k}{3\pi}$.

Substituting (18) in (16) and equating the coefficients of sin pt and cos pt to zero, we get

$$\left.\begin{array}{l} a^2\dfrac{d^2v}{dx^2}+p^2v+\alpha p^2u^2=0 \\ a^2\dfrac{d^2v}{d^2x}+p^2v+\alpha p^2u^2=0 \end{array}\right\} \qquad (19)$$

The approximate solutions of (19) are

$$u_1= A_1 \sin px/a - B_1^2\alpha/2+B_1^2\alpha/6 \cos 2\, px/a+\ldots$$

$$v_1= B_1 \sin px/a - A_1^2\alpha/2+A_1^2\alpha/6 \cos 2\, px/a+\ldots$$

from $x=0$ to $x=b$.

$$\because y=R_1 \sin (pt+\phi x)$$

where $\qquad \tan \phi_x=\dfrac{B_1 \sin px/a-A_1^2\alpha/2}{A_1 \sin px/a-B_1^2\alpha/2}=\dfrac{f_1}{f_2}$ approx.

and $\qquad R_1^2=f_1^2+f_2^2.$

From the conditions that at $x=b$, $y=\gamma$ sin pt the values of A_1 and B_1 are determined. Let us now take the important case in which the period of the forced vibration equals the natural period of the string; then sin $pb/a=0$ and all the expressions are much simplified. The expression for the displacement then comes out to be approximately

$$Y=\sqrt{\frac{2\gamma}{a}}\sin px/a\ .\ \sin(pt+\phi_1)$$

where $$\tan\phi_1=-\left(\frac{2}{\gamma a}\right)^{\frac{1}{2}}\sin px/a \tag{20}$$

From (20) we see that the amplitude of vibration of the string is proportional to the square root of the obligatory displacement unlike the case when friction is proportional to the velocity, where the amplitude of vibration is directly proportional to the obligatory displacement. The motion of all points on the string except near the nodes is in approximately the same phase. The phase of the motion at the nodes is the same as that of the obligatory motion, while that of the rest of the string lags behind by quarter of an oscillation. The phase changes rapidly near the nodes, and two points of the string on opposite sides of the node at a sufficient distance from it generally move in opposite directions.

The experimental results represented by fig. (4) agree with the theory according to which the maximum amplitude of vibration at a point for different values of γ increases much less rapidly than in proportion to it.

The effect of the finite amplitude of vibration can be easily calculated. The total length of the vibrating string at any instant is given by the formula

$$l=\int_0^l dx\left\{1+\left(\frac{dv}{dx}\right)^2\right\}^{\frac{1}{2}}$$

$$=\int_0^l dx\left\{1+\tfrac{1}{2}\left(\frac{dy}{dx}\right)^2\right\}$$

$$=l_0+\tfrac{1}{2}\int_0^l\left(\frac{dv}{dx}\right)^2 dx$$

Whatever may be the nature of the function $\left(\frac{dv}{dx}\right)$, the effect of the finite amplitude is to increase the length. If, however, there be no longitudinal motion, it produces an increase in tension which would amount to a large value when the modulus of elasticity is large. This increase in tension will therefore bring the points of maximum resonance towards lower tension and shift the whole resonance curve. This is the result actually obtained

as will be found from a reference to the experimental curve given in fig. (5).

Section V.—Theory of Forced Oscillations under Longitudinal Excitation.

We shall now pass on to the case of forced vibration maintained by forces of double frequency. The theory of oscillations under a periodic force varying the 'spring' was first given by the late Lord Rayleigh* on the assumption that the resisting force is proportional to the velocity. The theory has since been extended by Prof. C. V. Raman † who has shown how the equations may be modified and solved so as to give a determinate amplitude for the motion.

Case (I).—*Damping Proportional to the Velocity.*

This has been dealt with by Prof. Raman in the papers cited. The equation of motion of an oscillation of one degree of freedom under variable spring and subject to damping proportional to the velocity is

$$\ddot{U}+k\,\dot{U}+(n^2-2\alpha \sin 2pt+\beta U^2)\,U=0$$

The solution for the case in which the force has double the frequency of the maintained motion is

$$U=A_1 \sin(pt+\epsilon_1)+A_3 \sin(3pt+\epsilon_3)+A_5 \sin(5pt+\epsilon_5)+\text{ etc.}$$

where A_1 and ϵ_1 are given as a first approximation, by

$$(n^2-p^2+\tfrac{3}{4}\beta A_1^2)^2=\alpha^2-k^2p^2 \qquad (21)$$

and

$$\tan\epsilon_1=\left(\frac{\alpha-kp}{\alpha-kp}\right)^{\frac{1}{2}} \qquad (22)$$

Case (II).—*Damping Proportional to the Square of the Velocity.*

The equation of motion for this case is

$$\dot{U}\pm k\dot{U}^2+(n^2-2\alpha \sin 2pt+\beta U^2)U=0.$$

We assume the solution to be given by

$$U=A_1 \sin(pt+\epsilon_1)+A_3 \sin(3pt+\epsilon_3)+A_5 \sin(5pt+\epsilon_5)+\text{etc.}$$

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* Scientific Papers, Vol. II, page 188.

† Bulletin No. 6 of the Indian Association for the Cultivation of Science; also *Physical Review*, Dec. 1912.

Proceeding in the same way as before, we obtain the following relations as a first approximation

$$(n^2-p^2+\tfrac{3}{4}\beta A_1^2)^2=a^2-\frac{64}{9\pi^2}k^2A_1^2p^4 \tag{23}$$

$$\tan\epsilon_1=\left(\frac{a-\dfrac{8}{3\pi}kA_1p^2}{a+\dfrac{8}{3\pi}kA_1p^2}\right)^{\frac{1}{2}} \tag{24}$$

It is important to notice the points of agreement and difference between the results in the two cases.

The amplitude of vibration is not symmetrical with respect to n^2-p^2 in both the cases. In case (I), that is with damping proportional to the velocity, we find that the amplitude of vibration increases continuously as (n^2-p^2) diminishes. This continues with (n^2-p^2) negative, and equation (21) does not put any limit to the continuous decrease of (n^2-p^2) with consequent increase of the amplitude of vibration. It indicates that the tension can be diminished indefinitely, and the amplitude will then attain a correspondingly large value. But equation (23) of case (II) puts a limitation to the indefinite increase of the amplitude corresponding to an infinitely small tension. It shows that the amplitude of vibration is small when the tension is high, and increases with diminution of tension. The maximum value of the amplitude is not reached when $(n^2-p^2)=0$, i.e. when the natural frequency of oscillation is half that of the maintaining force. When (n^2-p^2) becomes negative (the natural frequency of oscillation of the string is less than half that of the maintaining force) the amplitude increases still further. But the rate of increase of the amplitude in case (II) is not so rapid as in case (I). This is due to the fact that the right side of (23) goes on diminishing with the increase of the amplitude whereas the right side of (21) remains constant. This goes on till the right side of (23) becomes zero. At this stage the motion collapses, since the supply of energy becomes less than that lost by dissipation. The maximum value of the amplitude is, therefore, obtained by putting the right side of (23) equal to zero; it is then given by

$$A_m=\frac{3\pi a}{8kp^2} \tag{25}$$

The same result may be obtained in another way. Solving equation (23) we have

$$A_1^2 = \frac{(l^2+4\beta^2 m)^{\frac{1}{2}} - l}{2\beta^2}$$

where $\qquad l = 2\Delta\beta + k_2$ and $m = a^2 - \Delta^2$

and $\qquad \Delta = n^2 - p^2$ and $k_2^2 = \dfrac{64}{9\pi^2} k^2 p^4$

The value of Δ when A_1^2 is maximum is given by

$$\frac{\partial A_1^2}{\partial \Delta} = 0 \text{ which gives } \Delta = -\frac{a^2\beta}{k_2}$$

The maximum value of A_1 is thus as before equal to

$$A_1 = \frac{3\pi a}{8kp^2}$$

Thus we find in case (II) there is a limit to the increase of the amplitude with diminution of $n^2 - p^2$ whereas in case (I), equation (21) does not put any limit to the increase of the amplitude.

The foregoing indications of theory agree closely with the observed facts. The continuous increase of the amplitude with diminution of tension has been found to be the same as required by the theory. The theoretical value of the amplitude when $n = p$ is the same as found from experiment.

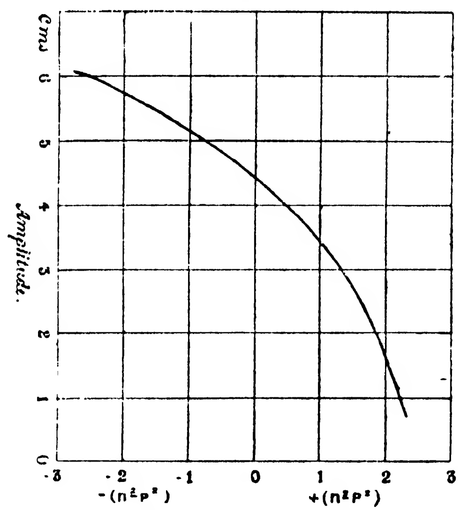

FIG. 7.—Resonance-Curve for Variable Spring of Double Frequency with Damping proportional to the Square of the Velocity.

The theoretical graph fig. (7) plotted from equation (23) shows the relation between the amplitude of vibration of the string and $n^2 - p^2$. The values of a and kp required for drawing the theoretical graph were determined directly from the known elasticity and damping of the string, and from the amplitude of the obligatory motion of the end of the string and the frequency of the motor vibrator. The agreement of the curves in figs. (6) and (7) is close. The amplitude is not symmetrical with respect to $n^2 - p^2$ as can be seen from the figures. The values of maximum amplitudes agree in the two curves. When $n=p$, the amplitude is given by the point where the curve cuts the axis representing the amplitude. Both the curves agree here also. The rate of increase of the amplitude is not the same everywhere in fig. (7): it is less near the collapse point than at small amplitudes of motion in accordance with the observed fact indicated by fig. (6).

Equation (21) of case (I) expresses the fact that in order to maintain the vibratory motion of the string, a must be greater than kp whatever the values of n and p, whereas equation (23) does not fix any minimum value for a. An attempt was made to maintain the motion of a long string with a very small magnitude of a, but it appeared that no maintenance was possible. This indicates that the frictional resistance for very small amplitudes is proportional to the first power of the velocity, and equations (21) and (22) then hold good; as soon as the amplitude of vibration of the string becomes sensible, the observed facts are better in agreement with the theory worked out on the basis that the damping forces are proportional to the square of the velocity.

The changes in the phase relation between the maintaining force and the maintained motion of the string may now be explained. In case (I), equation (22) shows that when a, k and p are constant, i.e. when the amplitude of the motor-vibrator, its frequency, and the frictional coefficient are constant, the phase angle ϵ should remain constant, whereas equations (23) and (24) of case (II) show that the phase angle will change even when the above quantities are constant if the permanent tension of the string and consequently its amplitude of vibration are altered. Equation (24) shows that as the tension is gradually decreased, at the initial stage where the amplitude of vibration is small, the phase angle varies very slowly, but when the amplitude is considerable, then it begins to decrease rapidly, and it continues

to diminish, and attains the limiting value when $A_1=\frac{3\pi a}{8kp^2}$, i.e. when the motion of the string collapses. This is in agreement with observation.

SECTION.—SYNOPSIS.

1. For large amplitudes of vibration it has been found that the frictional resistance to the vibration of a string in air is approximately proportional to the square of the velocity.

2. The forced vibration of a simple vibrator resisted by forces proportional to the square of the velocity has been dealt with, and it has been found (1) the maximum resonance is proportional to the square root of the impressed force, (2) the sharpness of resonance is directly proportional to the frequency natural to the system, and inversely proportional to the square root of the product of the frictional coefficient and the impressed force.

3. In the vibrations of a string obtaining by imposing a transverse obligatory motion at one point it is found that (1) the maximum amplitude of vibration increases much less rapidly than in proportion to the obligatory displacement. This has been theoretically accounted for by the square law of damping; (*b*) the effect of finite amplitude of vibration in shifting the resonance curve towards lower tensions has been shown by graphs and explained.

4. In the longitudinal type of Melde's experiment when the frequency of oscillation maintained is half that of the imposed variation of tension, it is found experimentally that (*a*) there is a definite limit corresponding to a particular magnitude of variation of tension imposed on the string, beyond which the amplitude of vibration cannot be increased by decreasing the constant part of the tension, the maintenance collapsing when this stage is reached and passed; and (*b*), that the phase relation between the maintaining force and the maintained oscillation varies in different parts of the range of maintenance even when the imposed variation of tension is kept constant in frequency and magnitude. These facts cannot be fully explained if the frictional force resisting the motion of the string is assumed to be proportional to the first power of the velocity, but agree with the theory developed on the assumption that the friction is proportional to the square of the velocity.

5. The experimental work described in the paper was carried out by the aid of the motor-vibrator designed by Prof. C. V. Raman, which was found specially suitable for the present investigation and indeed made it possible.

6. The author hopes on a future occasion to develop and present the fuller theory for the types of maintenance with other frequency ratios.

VI. The Magneto-Crystalline Properties of the Indian Braunites.

By K. Seshagiri Rao, B.A. (Hons.), Research Scholar in the Association.

CONTENTS.

SECTION I.—INTRODUCTION.

Braunite is a natural ore of manganese which according to Dr. Leigh Fermor, who has made an extensive study of the available deposits,* is next to psilomelane the most important of the Indian manganese minerals and occurs in great abundance in various parts of the country. It is also found in other parts of the world both in crystallised and in massive form.† In spite of the enormous quantities of this mineral that are available it is only comparatively rarely that it is found in India in the form of measurable crystals. The detailed account of such occurrences will be found in the memoir already cited. A feature of this mineral which is not mentioned by Dana and to which attention has been drawn by Dr. Fermor is that it is invariably slightly magnetic.‡ In view of the great importance of the mineral, it was thought that a quantitative study of its magnetic properties in the various occurrences in India would be of interest.

* Memoirs of Geological Survey of India, Vol. XXXVII.
† Dana's 'System of Mineralogy,' 1894 Edn., page 233.
‡ Page 63 *loc. cit.*

SECTION II.—PHYSICAL CHARACTER OF SPECIMENS EXAMINED.

So far specimens from two localities have been examined by the author.

The first specimen, which was very kindly furnished for the experiments by Dr. Fermor, was taken from the collection of the Geological Survey of India (No. K 348) and was from Kacharwahi in the Nagpur District in the Central Provinces. The specimen as received showed on the exterior a patch of white attached material which was a felspar and a small amount of decomposed blanfordite brown in colour. Except for these impurities, however, the whole specimen was part of one crystal showing well-cut faces of a deep steel-gray colour showing sub-metallic lustre, and when the specimen was ground down on a carborundum wheel to the shape of a sphere 2·4 centimetres in diameter, it was found that there was no visible felspar or blanfordite left on the surface. The sphere showed at eight regions on its surface (forming roughly the corners of a cube) a brilliant metallic glitter due evidently to the crystalline cleavages. It seemed thus quite certain that the sphere thus obtained was part of a homogeneous crystal. The material was hard and rather heavy, the density being 4·8. The Kacharwahi crystals belong to the tetragonal system.

The other specimen was part of a hard lump of very pure ore picked up on the spot from the well-known hill at Kándri, in the Central Provinces, which contains probably the finest body of manganese ore yet found in India and certainly one of the finest in the world. This is a fine-grained crystalline ore composed of apparently of braunite with a certain proportion of admixed psilomelane. A piece was broken out of this lump and also ground into a sphere of 2·4 centimetres diameter. It showed no visible impurity of any kind. The ore was very friable apparently owing to the admixed psilomelane. Its density was 4·22.

SECTION III.—EXPERIMENTAL METHODS AND RESULTS.

The method of experimenting used was to grind the specimen into a sphere and determine the force pulling it into a non-uniform field, and from this force the average value of intensity of magnetisation for the sphere was calculated. If the susceptibility is constant throughout the spherical specimen, then to a first approximation the intensity of magnetisation calculated from the

pull is the value of intensity at the centre of the sphere. This average value was the value that was determined. The advantage of the method is that one can easily test the homogeneity of the material. This may be done by simply inverting the sphere with reference to the field, in which case the pull will remain unchanged only if the lower and the upper half are alike. Within the limit of accuracy of this test the specimen was homogeneous. The expression for the force exerted upon an inductively magnetised sphere along the y-axis is given by

$$F_y = \left[I_x \frac{\partial H_x}{\partial y} + I_y \frac{\partial H_y}{\partial y} + I_z \frac{\partial H_z}{\partial y} \right] v.$$

where F=Mechanical force,
I=Intensity of magnetisation,
H=Magnetic field strength.

If the magnetisation lies in the xy-plane the last term drops out and if the direction of the x-axis be taken as that of the field at the centre of the sphere, the second term will have a very small value, so that the final simplified result becomes

$$F_y = I_x \frac{\partial H_x}{\partial y}.$$

To measure the force acting on the sphere, the following arrangement was adopted. A glass strip 2·5 cms. by 20 cms. was firmly clamped at one end to a rectangular brass piece fixed at right angles to a brass pillar moving up and down. The glass plate could be thus placed in proper position in the magnetic field. The other end of the plate carried a light wooden platform on which the sphere could be mounted. A pointer about 20 cms. long was attached to this end. In taking readings of the force the sphere was placed on the wooden platform. The bending of the glass plate as the field was slowly applied was measured by the motion of the end of the pointer which was observed through a microscope containing a scale in the eyepiece. The force acting on the sphere in dynes was calculated from the bending thus produced. For calibration the deflection produced by a known mass placed on the same position as the sphere was observed. With the arrangement used in the work a deflection of one division in the scale of the microscope eyepiece corresponded to about fifty dynes.

The magnet used in the work was a powerful electromagnet with large conical pole pieces which could be screwed into the iron cores of the magnet. These iron-cores moved in a groove in a strong iron bed plate and could be clamped in any desired position.

The field strength was determined by an exploring coil consisting of 40 turns and an equivalent area of 10·2 square centimetres. The coil was connected to a ballistic galvanometer in series with a suitable resistance. The galvanometer throw when the field was suddenly introduced was noted. The galvanometer was standardised by a condenser of capacity 0·2 microfarad.

The field gradient was determined by determining the fields at two points which were a definite distance apart.

Observations with the Specimen No. I.

The sphere was first hung at random in a magnetic field by a silk fibre. A directive twist was observed showing that the susceptibility of the crystal varied in different directions. Now when a crystalline sphere having different susceptibilities in different directions is placed in a uniform magnetic field, a restoring couple due to this difference and clearly proportional to it tends to twist the sphere so that the direction of maximum susceptibility coincides with the direction of the lines of force. This direction of maximum susceptibility was determined in this way:—

First a uniform magnetic field was obtained by replacing the conical pole pieces by flat ones. Two circles at right angles to each other were marked on the sphere. The sphere was hung at random in this field and turned in its own plane along these circles at regular intervals. The frequency of oscillation was observed at each position. The positions on the two circles at which it was maximum were marked. It was evident then that these were the positions at which the restoring couples were maximum since the frequency of oscillation increases with the magnitude of the couple. Since the restoring couples are proportional to differences of susceptibility, the planes perpendicular to the axes of rotation containing these two points would each clearly contain the axes of maximum susceptibility. Hence the intersection of these would give the direction of maximum susceptibility. If this were made the axes of rotation the direction of minimum would be easily obtained.

Tested in this way the sphere was found to possess a direction of minimum susceptibility. When it was mounted with this minimum as an axis of rotation the restoring couple became very small, showing the independence of magnetisation with direction in this zone. If the sphere were perfectly true it ought to rest easily in any position but as there were slight departures from complete sphericity and as the sphere could not be exactly placed in this position the couple did not absolutely vanish.

The direction of the maximum may be somewhat more quickly obtained by hanging the sphere mounted at random in a magnetic field since the direction of the maximum and that of the field should coincide. By repeating this process the plane of maximum susceptibility may be fixed.

The sphere was then mounted on the plate and placed in the non-uniform field produced by the conical pole pieces. Readings for the force were observed for different fields. Now with a constant susceptibility the force exerted must vary as $H \frac{\partial H}{\partial x}$, i.e. as H^2. Within the accuracy of measurements made it was found that both in the direction of the minimum axis and in the perpendicular plane that F/H^2 was constant, i.e. the substance was paramagnetic. The maximum field used was about 4000 gausses.

Field and field gradient were measured by galvanometer deflections. Field gradient was found nearly proportional to the field, i.e. $H = K. \frac{\partial H}{\partial x}$, and average value of K was used to calculate susceptibility. The value of susceptibility was thus found to be $0{\cdot}4 \times 10^{-3}$.

Readings for the force in the direction of axis of minimum and to the plane perpendicular to it were nearly identical. The difference of the susceptibilites was found to be too small to be exactly determined by the method adopted. An idea of the magnitude of this quantity was obtained as follows:

The moment of the couple tending to turn the sphere in a magnetic about the axis of x from y towards z is given by

$$L = \tfrac{4}{3}\pi a^3 (\gamma B - \beta C).*$$

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* Maxwell's Electricity and Magnetism, Vol. II.

When referred to principal axes of magnetism the expression becomes

$$L=\tfrac{4}{3}\pi a^8\frac{r_2-r_3}{(1+\tfrac{4}{3}\pi r_2)\,(1+\tfrac{4}{3}\pi r_3)}\,YZ$$

$$=\tfrac{4}{3}\pi a^8\,\frac{r_2-r_3}{(1+\tfrac{4}{3}\pi r)^2}\,F^2\theta,$$

since when $X=O$, $Y=F\cos\theta$, $Z=F\sin\theta$, provided θ is small.

Thus by observing the time of oscillation when the sphere was mounted in a uniform field with its axis of rotation in the magnetic plane of symmetry and measuring the strength of the field, $\frac{r_2-r_3}{r}$, or the maximum difference of susceptibility was determined in terms of the average susceptibility. This difference was found to be about 2% of the value of average susceptibility.

The magnetic properties of the crystal as investigated thus bear a close resemblance to its physical characteristics. Braunite belongs to the tetragonal system, having a plane of symmetry perpendicular to the vertical axis. The value of $\frac{1}{c}$ the vertical axis is given by Dr. Fermor as * 0·99. Magnetically the crystal has been found to possess a plane of symmetry at right angles to which is the axis of minimum susceptibility, the difference of susceptibility being of the same order as that of the difference of crystallographic axes.

Specimen No. II.

As the specimen No. II was not a part of a single crystal but only a homogeneous mixture of braunite and psilomelane, the determination of its susceptibility only was made. The observations were made by comparison with those of specimen No. I. This was also found to be paramagnetic as in the case of No. I, but its susceptibility was only 75% of that of No. I.

It would be interesting in view of the interesting magnetic property possessed by Indian braunites, to make a detailed investigation of European braunites which as far as one could gather from the literature have not received any attention.

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* Loc. cit. page 56.

Section IV.—Chemical Composition.

Both of these specimens were analysed in the chemical laboratory of the Association. The results of these analyses are shown below :—

	Kacharwahi.	Kandri.
Iron	2·5%	2·6°
Manganese	54%	51·4
Silica and insoluble residue	11·8%	12·6

Both the specimens contain a relatively low proportion of iron.

Analysis of these specimens as recorded by Dr. Fermor are also given below for comparison.

	Kacharwahi.	Kandri.
Manganese	57·86	58·55
Iron	3·85	3·37
Density	4·79	4·28

It seems not unlikely that the magnetic property of braunite may be largely due to its manganese content, though on account of the presence of iron oxides in the substance it is difficult to pronounce decidedly on this point at present.

Section V.—Synopsis.

The results of investigations upon the magnetic properties of Indian Braunites may be stated as follow :—

(1) Symmetry of braunite is magnetically that of an axis of symmetry, with a plane perpendicular thereto in which the magnetic susceptibility is independent of orientation.

(2) Along this axis of symmetry the susceptibility is minimum and in the plane it is maximum, the difference between the two being about 2%.

(3) Both in this plane and along the axis it is paramagnetic, the susceptibility being very low.

(4) All these magnetic properties bear a close resemblance to the physical characteristics of the crystal which are a crystallographic plane of symmetry at right angles to the vertical axis, the value of the axis $\overset{1}{c}$ being 0·99.

In conclusion, the writer wishes to thank Professor Raman who suggested the investigation and took much interest in its progress.

VII. Free and Forced Convection from Heated Cylinders in Air.

By Bidhubhusan Ray, M.Sc., Palit Research Scholar in the University of Calcutta.

(Plate VI).

CONTENTS.

SECTION I.—REVIEW OF PREVIOUS WORK ON CONVECTION.

The phenomenon of convection from the surface of a heated body immersed in a fluid has received considerable attention both from the theoretical and the experimental points of view. If the cooling fluid be a gas, then, the variations of pressure, density, velocity, thermal conductivity and viscosity at different points, so complicate the problem that not only has little progress towards a complete mathematical solution yet been made, but for some time a true knowledge of its laws seemed nearly impossible. A. Oberbeck * gave the general differential equation for this problem, but being unable to solve it generally was satisfied with working out a solution for a special case. Later on, L. Lorentz † obtained a solution for the case of a vertically placed plane strip cooling in air. When the strip is protected from draughts, he found that

* Ann. Phy. 7, 271 (1879).

† Ann. Phy. 13, 582, 1881.

the heat convected away from the surface varies as $\theta^{5/4}$, where θ is the difference of temperature between the strip and air before it is heated by the strip, and this value agreed fairly well with some of the older experimental work. H. Wilson* attempted to solve the problem taking the viscosity into account but his solution is limited to a very special case.

In 1901 Boussinesq † took up the problem and first clearly stated the laws for the forced convection of heat for a body immersed in a stream of liquid when the flow is not turbulent. The same author after four years published a memoir containing a great number of successful solutions of the problem of the convection of heat from bodies of various shapes immersed in a stream of fluid. It is to be noticed here that according to Boussinesq's result, the heat loss due to convection of heat from bodies immersed in a stream of fluid varies as $V^{\frac{1}{2}}\theta$, where V is the velocity of the current and θ is the difference of temperature of the solid body above that of the surrounding fluid at a great distance. Kennelly,‡ making elaborate measurements of the convection of heat from thin copper wires, remarked as follows:—"The lateral conduction through air is negligible because the air does not remain at rest but expands and flows convectionally. Consequently we may safely ignore conductive thermal loss. Convection loss from the air is a hydrodynamical phenomenon, involving the flow of air past the surface of the wire and the amount of heat which the moving stream can carry off. Very little seems to be known quantitatively about convection." Being unable to find any general solution, he was therefore satisfied with framing an empirical law to express his results. In the case of forced convection of heat his results agreed with those of Boussinesq for the cooling of cylindrical wires.

Russel,§ following after Boussinesq, discussed the problem at some length. In order to solve the general differential equation and apply its results to bodies of various shapes and dimensions, Russell was compelled to make certain assumptions with regard to the nature of the fluid sorrounding the solid. The liquid is

* Camb. Phil. Soc. Progs. 12, 406, 1902.
† Comptes. Rendus. Vol. 133, p. 257.
‡ Trans. Amer. Inst. E.E. 28, 263, 1909.
§ Phil. Mag. 20, 591, 1910.

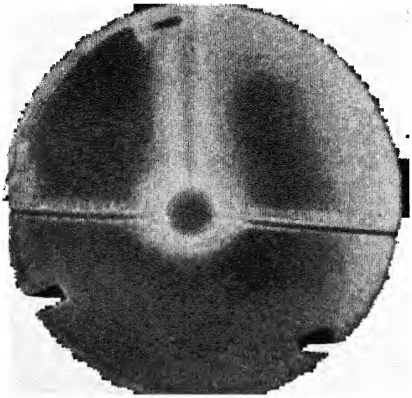

Fig. 1

Free Convection

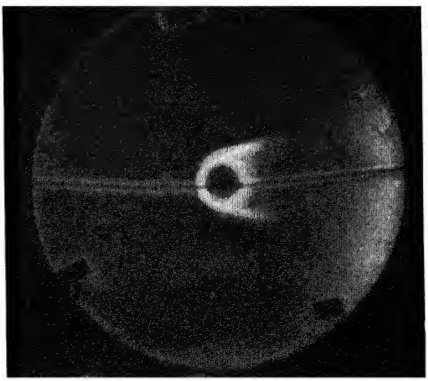

Fig. 2

Forced Convection

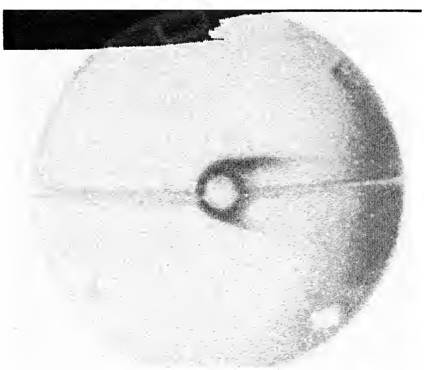

Fig. 3

Forced Convection

supposed to be opaque to heat rays, it has no viscosity and so it slips past the surface of the solid, also it is supposed to be incompressible. The further assumptions are, "The thermal conductivity of the liquid is very small and the variations in its density does not appreciably alter the trajectories of the liquid particles in the immediate neighbourhood of the solid from the shape they have during isothermal flow. The former assumption is true in most practical cases, and the latter is permissible when the velocity of the current is appreciable and no eddies are formed. The surface of the solid being cooled by the current is supposed to be isothermal, and the liquid in immediate contact with it at any instant is supposed to have the same temperature as the solid." In view of these very important assumptions we should expect the solutions to give only approximate values when applied to the problem of spheres and cylinders cooled by currents of air. The result, which Russell arrived at, and which is true for the two-dimensional flow round a solid of any shape immersed in a stream of liquid, agrees with that of Boussinesq; and when particular cases are considered it agrees with the work of Kennelly,* Ayrton and Kilgour † and other workers. All the investigators in calculating the heat loss from the surface of the body find it is proportional to the difference of temperature between the solid and liquid surrounding it. It is to be noted that Newton's law of cooling is thus verified when the cooling fluid is a liquid, and Newton himself enunciated his law with reference to the convection and not the radiation of heat. In several practical applications, Newton's law of cooling leads to results agreeing closely with experiment. In the theory of a hot-wire oscillograph Irwin ‡ has assumed that the convection loss of heat from a heated strip immersed in oil is proportional to the difference of temperature between the metal and the oil. The very satisfactory coincidence between the theoretical and experimental result shows that the assumption is approximately correct.

Russel in the same paper shows that in the case of turbulent motion of water through a pipe, Newton's law is very approximately true. It is to be noted here, that we are now discussing

* Journ. Inst. Elect. Engin. 35, 364, 1905.
† Phil. Trans. Vol. 183, Pt. i, p. 371, 1892.
‡ Journ. Inst. Elect. Engin. 39, 617, 1907.

the problem of forced convection of heat, but in the case of free convection Newton's law of cooling is not applicable in liquids, where Lorentz's law that the heat loss in proportional to $\theta^{5/4}$, where θ is the difference of temperature between the strip and air before it is heated by the strip is applicable.

I. Langmuir * differed from the previous workers in two of the important assumptions, namely in the effect of viscosity and the variation of the thermal conductivity due to temperature. This author contends that in the case of convection from small wires the effect of viscosity is most important. His own views are clearly given thus in his thesis on some reactions around glowing Nernst filaments: "According to the kinetic theory, the viscosity of gas increases with square root of the absolute temperature, the driving force of convection being proportional to the difference in density between the hot and cold gas increases only very slowly with increasing temperature. Therefore in the immediate neighbourhood of the filament the flow of gas is small and the heat must be carried away practically only by conduction. It is highly probable that at a very high temperature, the motion of a gas in the immediate neighbourhood of the wire would not perceptibly increase but probably decrease, while at the same time the heat conductivity of the gas would increase very greatly. Thus at a distance very near the surface of a body, heat conduction is more important than convection; so that heat will be carried only by radiation and conduction but not by convection." Several considerations had made it seem probable that the above theory would be fairly close to the truth. For instance it was found that the watts loss from a wire was the same, whether the wire was placed horizontally or vertically. Now the lines of flow of the heated air around the wire would be totally dissimilar in these two cases. Yet it was found that the energy necessary to keep a piece of pure platinum at a given temperature—the resistance being kept constant—never differed appreciably for horizontal or vertical wires, and at a bright red heat or above, the difference became negligibly small. This was a strong indication that the heat loss was dependent practically on heat conduction very close to the filament and that the convection current had practically no effect except to

* Phys. Rev. 34, p. 401, 1912.

carry off heat away after it has passed out through the film of adhering gas. The thickness of the film of gas through which conduction takes place can be calculated if the temperature of the wire, its diameter, and the heat conductivity of the gas are known. The last factor varies very much with temperature.

Langmuir, in calculating the heat loss from a heated wire, assumes that a thin cylindrical film of gas adheres to the wire through which heat travels only by conduction and radiation, and the thickness of the film of air is the distance heat must travel before the heat flux due to temperature difference becomes small compared with that due to convection. Throughout this film of air the viscosity, the thermal conductivity and specific heat of air varies greatly. Meyer * has given a formula for conductivity in terms of viscosity and specific heat (per gram) at constant volume, while Sutherland obtained an equation connecting viscosity and temperature containing two constants, which have been determined by Rankine.† M. Pier ‡ has obtained an equation for specific heat at constant volume in terms of temperature, and so connecting these formulae, Langmuir obtained a relation between conductivity and temperature, and applying the simple conduction formula in the thin film, he calculated the heat loss from the heated wire. His experimental result agrees fairly well with the theoretical formula.

Later on, the same author § published an account which is an extension of his earlier work to cover convection of heat from plane surfaces. Experiments are described in which the heat loss from plane surfaces at various temperatures is determined. After discussing the results of this experiment with his previous theory, he finds that the film theory is found not to apply to convection forced by air currents, but Russell's formula agrees well with the experimental data. Morris || has verified the application of the formula of the type obtained by Boussinesq to the cooling of fine wires heated by electric current to temperatures about 70°c above the surrounding air and for air velocities as high as 40 miles per hour.

Prof. L. V. King ¶ follows Boussinesq and transforms the

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* Kinetic Theory of Gases. † Proc. Roy. Soc. Lon. A84, 181, 1910.

‡ Z. f. Electrochem. 15, 536, 1909 and 16, 899, 1910.

§ Amer. Electrochem. Soc. Trans. 23, p. 299, 1913.

|| Electrician, Oct. 4, 1912; also Engineer, September 27, 1912.

¶ Phil. Trans. A, Vol. 214, 1914.

general equation of conduction into a greatly simplified form in the case of a liquid flowing past a cylinder, reducing the problem to the case of a two-dimensional flow. The problem can now be transformed into a form having a known partial differential equation. The complete solution of this equation requires a knowledge of the conditions of heat transfer over the interface between the solid and the liquid. "The solution of this problem which gives results in best agreement with experiment for the case of convection of heat from small cylinders is that obtained by assuming the flux of heat to be constant over the boundary. As a result of high heat conductivity of the cylinder in the experiment carried out, the temperature of the cylinder may be considered constant over the boundary and there will therefore be a discontinuity in the temperature of the boundary; we assume that the temperature of the stream in contact with the cylinder becomes finally equal to that of the cylinder at the point where it leaves the boundary." The result he obtained for the loss of heat for small and large velocities of the air current agrees fairly well with experiment and with that of some of the previous workers in the case of forced convection. It is to be noted here that King based his theory on simple hydrodynamical flow in which there is slipping at the boundary between solid and liquid. Here he takes no account of viscosity, variation of thermal conductivity and specific heat near the surface. It is interesting to note that he gets the value of some constants which agrees fairly well with that of the other investigators working on different lines.

Kennelly * later on found, that for pressures between 5 and 3 atmos., the rate at which heat is dissipated varies not only as the square root of the speed of the wire as already found out but also approximately as the square root of the air pressure in accordance with the theoretical result of Boussinesq.

V. M. Schidlovski † in testing the truth of L. Lorentz's ‡ formula for the case of cooling of a wire finds that the current sufficient to heat the vertical wire to redness leaves the horizontal one dull, and the effect due to conduction and convection is less in the case of vertical wire than when it was horizontal, and this

* Amer. Phil. Soc. Proc., Vol. 53, 55, 1914.

† Russian Physico-chem. Soc. J., 43, 132, 1914.

‡ Ann. Phys. 13, 582, 1881.

decrease is due to the ascending hot-air currents enveloping the wire.

T Barret *, while experimenting on radiation and convection from a heated wire in an enclosure of air, finds that as the temperature is raised the heat loss due to convection increases more rapidly than that due to radiation.

I. A. Hughes † has measured the loss of heat per c.m. length and also the loss at different velocities in the case of cooling of cylinders in a current of air, his results agreeing with those of the previous workers.

W. P. White ‡ has made an investigation to obtain data for the use in the design of calorimeters. According to him, leakage due to conduction and radiation conforms substantially to Newton's law of cooling. Heat transfer by convection, being due to air currents, whose temperature and velocity are both affected by thermal head, is more nearly proportional to the square of the temperature difference. The observed variation from Newton's law of cooling is to be ascribed to convection. He appears to have worked with cooling of bodies with small air gaps, so his result is not comparable with other workers who have worked with an infinite air gap. The results obtained both for eddyless motion and for turbulent flow are intended to serve as a practical guide in calorimeter designing.

In view of the fact that different (sometimes opposite) assumptions have been made by different investigators regarding the nature of the boundary conditions at the surface of the solid, and that there is much regarding the subject which is still obscure, the problem was taken up to study more fully

(1) the exact nature of the convection phenomena, and

(2) the temperature and velocity distribution round the heated body, and to find the law of flow both for free and forced convection of heat and to obtain material for dealing with the problem on a stricter hydrodynamical basis.

Section II.—Study of Convection by the Foucalt Method.

Some idea as to the nature of the convection phenomena may be formed by using smoke or some such substance round the

* Lond. Phys. Soc. Proc. 28, p. 1, 1915.

† Phil. Mag. 33, 118, 1916.

‡ Phys. Rev. 10, 743, 1917.

heated body, and allowing strong light to fall on this smoke which becomes luminous and makes the nature of its movement visible to the eye. A much better method which has been used by the present author is an optical method of observation which is a modified form of the Foucalt test or Topler Schlieren apparatus. According to the simple laws of geometrical optics, rays issuing from a point can be focussed at another point if the optical instruments are perfect. An eye situated just behind the focus observes an illuminated field, but if a sharp-edged screen be gradually advanced in the focal plane, all light is suddenly cut off and the entire field becomes dark simultaneously. At this moment if there be any irregularity in the optical surfaces or in the medium through which the rays come to a focus, the rays are deviated from their proper course so as to escape the screening and the field becomes luminous. From the picture of the luminous patch the nature of the irregularity of the optical appliances or of the medium through which the rays are passing can be is studied.

The arrangement of the experiment is as follows: A copper cylinder of radius 7·2 millimeters is used for the experiment. A heating coil made of nickel-chromium wire, wound on a frame of mica, similar to that used in platinum-resistance thermometers, is inserted into the cylinder, which is kept in a horizontal position by means of two stout wires as seen in the photograph. A current of 1·8 amp. is passed through the coil, and the cylinder which was insulated is thereby heated.

White light from a small circular hole, illuminated by the electric arc, is incident upon a good achromatic lens, which forms a sharp image of the hole at a considerable distance from the lens. A thin circular metal disc fixed on a thin sheet of mica is put in the focal plane, and the whole arrangement is so adjusted that the metal disk just completely cuts off the image of the circular hole formed by the achromatic lens. A telescope is placed with its object-glass just behind the metal disk attached to the sheet of mica and is pointed towards the lens. The copper cylinder from which the convection is observed is placed immediately in front of the lens with its axis along the optical axis of the system, and the observing telescope is focussed on the end of the cylinder.

When the cylinder is heated by passing the electric current,

the field of view of the telescope which is previously just moderately dark soon begins to show the luminous streams of air rising from the cylinder. The phenomenon seen gives a vivid visual picture of the nature of the temperature distribution round the cylinder and of the motion of the air.

Figure 1 in Plate VI illustrates *free* convection from the heated cylinder. The luminous patch seen is very broad and diffuse, and quite steady. The horns pointing upwards show the path along which the heated air rises. It will be seen that some distance above the cylinder and between the two horns we have a distinct darker vertical strip in which the temperature appears to be less than within the horns.

Figure 2 in Plate VI shows the effect observed when a gentle current of air from an electric fan is forced horizontally at right-angles to the cylinder. The horns of the luminous area are here much better defined, and it was noticed that they were somewhat unsteady in position owing to the fluctuations in the current of air. As visually seen the thin tips were much more pointed than would appear from the photograph. The photograph was taken with quite a short exposure. The luminous patch here is much sharper and was visually observed to be far more brilliant than in the case of free convection, showing that the gradient of temperature is greatly increased by the forced draught.

Figure 3 in Plate VI also shows a case of forced convection, the velocity of the draught being greater here than in the case of Fig. 2. The features mentioned above, are here seen in more accentuated form. A considerable difference in the thickness, brightness and sharpness of the luminous fringe in front of and behind the cylinder will be noticed.

In the photograph reproduced in the Plate, one of the luminous horns is seen to be distinctly brighter than the other. This is due to the fact that the circular metal disk which cut off the image of the source was not exactly symmetrically placed. It was very interesting to watch the luminous streamers on the ground glass of the camera when the position of the disk was slightly altered or when irregular draughts of air were allowed to fall on the cylinder. The adjustment of the disk to the final position of symmetry had to be made by actual trial. In studying the free convection, it was found necessary to protect the cylinder

from irregular draughts by surrounding it by a large wooden box.

Observations were then made of the diameter of the luminous fringe surrounding the cylinder and of the velocity of the wind. It was found that as the velocity of the wind was increased the diameter of the patch decreases at first very rapidly but after a time very slowly until finally with the increase of the velocity of air, it is difficult to detect any change in the width of the luminous patch. The following measurements of the diameter of the patch give an idea of the effects observed :—

Velocity of the wind.	Angle width of patch in the field of the observing telescope.
Free convection	$0^0-14'-44''$
5·6 ft. per min.	$0^0-12'-30''$
8·5 ft. per min.	$0^0-11'-28''$
9·8 ft. per min.	$0^0-9'-43''$
12·5 ft. per min.	$0^0-8'-47''$
18·2 ft. per min.	$0^0-8'-10''$

SECTION III.—ELECTRICAL DETERMINATION OF TEMPERATURE DISTRIBUTION ROUND THE CYLINDER IN FREE CONVECTION.

To confirm the general indications furnished by optical observation, the distribution of temperature in the region surrounding the cylinder was studied by an electrical method. A thermocouple of fine copper and constantan wires was used, the junction between them being made by electrical fusion. Care was taken to see that the junction of the two wires occupied a very small region. One end of the couple was placed in ice and the other placed near the cylinder. The position of this end with respect to the distance and direction from the cylinder could be determined by means of a telescope having a millimeter scale in the eyepiece. The thermocouple was balanced against a calibrated potentiometer bridge wire

and the balance point carefully noted. The cylinder was protected by a wooden box enclosing it, and very great care had to be exercised to avoid disturbing currents of air.

Five sets of readings for each position of the couple were taken and the mean of the five was used for drawing the form of the isothermals round the cylinder.

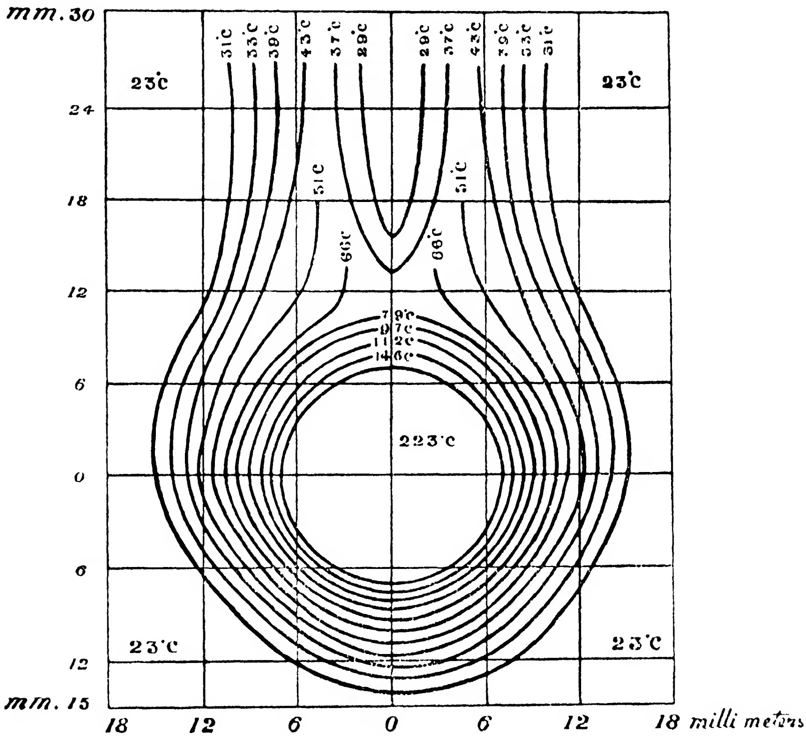

FIG. 4.—Temperature Distribution around Heated Cylinder.

The curve (Fig. 4) shows a striking agreement between the temperature distribution round the heated cylinder as observed electrically and as indicated by optical observation and illustrated in Fig. 1 in Plate vi.

It is to be noticed that the junction of copper-constantan occupies a finite space and hence when it is placed near the cylinder, the flow of heat round the cylinder is disturbed by its very presence. Some heat will be carried along the wires away

from the junction and some heat will be radiated away from the cylinder to the junction. These effects cannot be entirely eliminated but may be diminished by making the junction occupy a very small space and having the wires of the couple made very thin.

One important point to be noticed is that the isothermals are crowded together on the lower side of the cylinder, and diverge on the upper side of the cylinder.

SECTION IV.—SYNOPSIS AND DISCUSSION OF RESULTS.

The present paper describes the results of an experimental study of the free and forced convection of heat from horizontally held cylinders in air. The general character of the temperature distribution round the cylinder was optically observed by the method of the Foucault test and quantitatively determined by thermo-electric measurement of the temperature of the air. The following are the principal features observed :

1. In *free* convection, there is a relatively large region of heated air surrounding the cylinder, the temperature gradients are relatively small, and the region above the middle of the cylinder is actually cooler than the region vertically above its two edges. Immediately on the surface of the cylinder itself, the gas appears to be quiescent, the movement increasing as we pass away from the cylinder on either side. There is probably no actual discontinuity of temperature at the boundary itself, but there is a rapid fall of temperature as we move away from the surface.

2. In *forced* convection, the layer of heated air is much thinner, and the temperature gradients are accordingly greater. There is a noticeable difference in the thickness of the heated fringe of heated air on the leeward and windward sides of the cylinder, and this thickness decreases with increasing velocity of the air current. The 'horns' or currents of heated air moving away from the cylinder are much sharper in forced convection than in free convection.

3. The optical observations and the experimentally determined form of the isothermals in the fluid surrounding the cylinder suggest the question as to how far the hypothesis of King that there is a discontinuity of temperature between solid and the surrounding gas represents actual fact. The evidence suggests

rather that in the case studied by the present author that there is little or no slipping between the fluid and the solid at the boundary, and no actual discontinuity of temperature though there is a rapid fall of temperature close to the boundary. Though there is no actually stationary film of *finite* thickness of gas in contact with the solid, there appears to be very little motion in the immediate neighbourhood of the boundary, and presumably at the boundary itself the gas is at rest. By considering the effect of viscosity of the fluid, it would seem possible to explain the effects observed. The mathematical theory of convection contained in King's paper appears to require modification taking the effect of viscosity into account to enable the observed form of the isothermals to be explained, and the agreement obtained between King's theory and the observed convection-losses of heat from small cylinders may require to be interpreted afresh.

The experimental work of finding the distribution of temperature round the cylinder was carried out in the Palit Laboratory of Physics, and the author offers his best thanks to Prof. C. V. Raman, Palit Professsor of Physics, for suggesting the problem and for his lively interest in the progress of the investigation.

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1920.

VIII. Experiments on Impact.

By A. Venkatasubbaraman, B.A.

In a paper which has appeared in the Physical Review for April 1920, Prof. C. V. Raman has shown how the well-known theory of impact developed mathematically by Hertz may be extended and applied to the problem of the transverse impact of a sphere on a plane elastic plate of finite thickness. The theory involves the consideration of the proportion of the kinetic energy of the impinging body transformed into energy of wave-motion in the elastic plate. Starting with the assumption that, to a first approximation, the duration of impact is the same as on Hertz's theory for the case of an elastic plate of infinite thickness, the potential and kinetic energies carried off by the flexural waves generated in the plate by impact are calculated on certain simplifying assumptions regarding the form of these waves. The result finally obtained is that, e, the coefficient of restitution of the impinging body is given by the formula

$$e = \frac{f\rho a^2 - 0{\cdot}56\, M}{f\rho a^2 + 0{\cdot}56\, M}.$$

where

$$a^2 = \pi T f \sqrt{E/3\rho\,(1-\sigma^2)}$$

T is the duration of impact,
E is Young's Modulus for the plate,
σ is Poisson's ratio for the plate,
$2f$ is the thickness of the plate,
ρ is the density of the material of the plate,

and M is the mass of impinging body. If T (the duration of impact) be taken to have the value given by Hertz's theory for a plate of infinite thickness, we may expect a good agreement between the formula and the observed coefficient of restitution for

moderately thick plates. For *thin* plates, it would be more nearly correct to take T as the actual duration of impact which would no doubt differ somewhat from that for the case of an infinitely thick plate.

To test the foregoing formula, the author has made a series of observations using a set of polished hard steel balls of various diameters and a set of glass plates of different thicknesses. These materials were chosen as, provided the size of the impinging sphere is not too great and the velocity of impact is not too large, no permanent deformation of the impinging bodies results from the impact. For velocities of impact greater than a certain limit depending on the size of the impinging sphere, the collision results in the formation of percussion-figures of beautiful geometrical form which have been observed and described by Professor C. V. Raman ('Nature,' Oct. 1919). If a percussion-figure be formed, we cannot naturally expect the formula for the coefficient of restitution to remain valid, and it is accordingly necessary to restrict the comparison with it to cases in which the spheres and the velocities of impact are moderately small.

The accompanying table gives the results for the coefficient of restitution obtained by the writer for a certain moderate velocity of impact. The coefficient of restitution calculated from the formula are also shown, T being taken in the formula to be the duration of impact as given by Hertz's theory for an infinitely thick plate.

The following facts emerge on an examination of the figures shown in the Table of results. For the thicker plates, the experimental values for e are smaller by two or three per cent. than the theoretical values. This is evidently due to various minor causes of dissipation of energy not being taken into account in the theoretical treatment. For moderately thick plates, the calculated and observed coefficients of restitution agree well. Theory and experiment also agree in the case of very thin plates in giving a zero coefficient of restitution. In other words, in such cases, the sphere on impact with the plate remains in contact with it and behaves nearly in the same way as a perfectly inelastic body. But in certain intermediate cases where the thickness of the plate is less than about half the diameter of the impinging sphere but not so small as to give a zero coefficient of restitution, the observed

COEFFICIENT OF RESTITUTION.

Velocity of Impact 234 cms. per second.

Thickness of Plate in Centimetres.	Diameter of Spheres in Centimetres.													
	0·791		0·714		0·637		0·555		0·396		0·314		0·237	
	Calc.	Obsd.	Calc.	Obsd.	Calc	Obsd.	Calc.	Obsd.	Calc.	Obsd.	Calc.	Obsd.	Calc.	Obsd.
2·53	0·98	0·95	0·99	0·96	0·99	0 97	0·99	0·97	0·99	0·98	0·99	0·98	0·99	0·98
1·93	0·97	0·95	0·98	0·96	0·98	0·96	0·99	0·97	0 99	0·97	0 99	0 98	0·99	0·98
1·29	0·93	0·91	0·94	0 93	0·96	0·93	0 97	0·94	0·98	0 95	0 99	0 97	0·99	0·98
1·01	0·89	0·87	0·91	0·89	0·93	0·91	0·95	0·93	0·97	0·94	0·98	0·96	0·99	0·97
0·71	0·79	0·77	0·83	6·80	0·86	0·83	0·89	0·86	0·94	0·90	0·96	0·95	0·98	0·96
0·49	0·61	0·60	0·67	0·66	0·73	0·71	0·79	0·77	0·89	0·86	0.93	0·91	0·96	0·94
0·35	0·36	0·38	0·44	0·45	0·53	0·53	0·62	0·61	0·79	0·77	0 86	0·84	0·92	0·89
0·225	0	0	0 03	0·13	0·15	0·23	0·28	0·34	0·55	0·55	0·69	0 68	0·81	0·79
0·150	0	0	0	0	0	0	0	0·12	0·21	0·28	0·42	0·45	0 62	0 63
0·105	0	0	0	0	0	0	0	0	0	0·08	0·09	0·24	0·36	0·43

values of e are somewhat larger than the calculated values. This appears to indicate that in the case of such plates, the actual duration of impact is somewhat greater than that given by Hertz's theory for the case of an infinitely thick plate, and that better agreement between theory and experiment would be obtained if such actual duration was taken as the value of T in the formula. (Of course, this would only be permissible provided the actual duration is finite and not very much greater than given by Hertz's formula.)

Some observations made by the writer show that when the velocity of impact on the plate is so great as to result in the formation of percussion-figures, the coefficient of restitution falls considerably below the value given by the formula above. This is what may be expected in view of the fact that the production of the percussion figures must result in dissipation of energy which involves a decrease of the coefficient of restitution. In this connection, it is also worthy of note that much higher falls and therefore larger velocities of impact may be used in the case of small spheres than in the case of larger spheres without resulting in production of percussion-figures. As the magnitude of the stresses per unit area set up by impact is (according to Hertz's theory) the same for the large as well as foı small spheres, this observation suggests that the duration of impact is distinctly a factor in determining whether or not the impact should result in the production of a percussion figure. The author has also observed that the area of the percussion-figure increases appreciably with increasing velocities of impact, and also, as might be expected, with increasing size of the impinging sphere. From the observations of the decrease of the coefficient of restitution, it is possible to calculate the energy required for the production of the percussion-figure and to find how it varies with the area of the internal fracture which constitutes the percussion-figure.

The author has also found that appreciable deformations from planeness of the surface of the glass plate result from the production of percussion-figures and may be detected by optical methods. This phenomenon is no doubt intimately connected with the question of residual strains in the plate left after production of the internal fracture.

IX. The Theory of the Flute.

By Dr. G. T. Walker, C.S.I., Sc.D., F R.S., Director-General of Observatories in India.

Among the more attractive applications of mathematical physics are those which have been made to musical instruments; and although in most cases the main principles have been truly laid down there are details still unexplained which may be of some theoretical or practical interest. Some recent experience in designing piccolos of different bores has brought out various points in which the existing theory of the flute appears incomplete, and I propose to give a short account of these in the hope that it may stimulate a further theoretical examination of the subject. An excellent account of the many papers dealing with the acoustics of pipes will be found in Winkelmann's Handbuch der Physik (Vol. II, pp. 446, 7. Leipzig, 1909).

2. The questions at issue fall naturally into four classes as they are concerned with (*a*) the shape of the bore of the instrument; (*b*) the position of the holes and open end; (*c*) the size of the holes; (*d*) the method of blowing; and (*e*) the method of fingering. Considering first the bore, Boehm pointed out that while a flute with a large internal diameter possessed facility and tone in the lowest notes the highest notes were unmanageable; and $\frac{3}{4}$ of an inch or 19 mm. is generally accepted as the standard for the ordinary *D* flute whose length is about 24 inches. For the ordinary *D* piccolo a bore of $\frac{3}{8}''$ gives a rather feeble lowest octave and the usual bore of $\frac{7}{16}''$ is better: one of $\frac{1}{2}''$ is brilliant in its lowest notes, but rather difficult in the highest. Boehm also pointed out that a flute with a purely cylindrical bore cannot be in tune throughout its range, the notes of the second octave being flat by comparison with those of the first and those of the third by comparison with those of the second. It would thus

appear that the correction for the open end or for a large hole in a cylindrical tube increases appreciably with the frequency of the note.[1] Boehm rightly gave as the remedy for this difficulty a constriction of the bore of the upper portion of the tube (that containing the mouth-hole) so that the bore diminishes gradually from its full dimensions at a quarter of the length from the top end to nine-tenths of the dimensions at the mouth-hole. The same result is apparently produced in a conical flute by making the top third cylindrical and the lower two-thirds conical, the diameter at the open end being about ·7 of that of the cylindrical portion.

3. The length of the tube to produce a given note will differ from the half of the corresponding wave length by the sum of the corrections for the mouth-end of the flute and for the open end. Similarly the positions of any of the holes will depend on the sum of the corrections for the corresponding hole and for the mouth-end. An obvious way of determining the facts is to measure the distances of a series of holes of equal size from the mouth-hole in an actual flute. When the unknown but constant correction has been subtracted these distances must bear to the unknown distance for the fundamental note a series of known ratios depending on the musical intervals of the notes in question. We thus have a number of equations with only two unknowns, these being the constant correction (the sum of the corrections for the mouth end and for a hole of the given size) and the distance for the fundamental note. In this way I have deduced from my flute, with a bore of $\frac{3}{4}''$, that the distances of the $\frac{1}{2}''$ holes from the mouth-hole are less by 2·40″ than their position if there were no corrections; for a $\frac{5}{8}''$ hole the corresponding amount is 2·27″. Also the length from the mouth-hole to the open end is less by 2·0″. Now when the bore is 1/30 of the wave length Bosanquet (see page 425 of the volume of Winkelmann above referred to) found a correction for the open end of ·543 of the radius; and for a bore of 1/12 of the wave-length he found ·635. For the lowest note *C* of my flute the bore is 1/34 of the wave-length, and I estimate that the correction for the open end should be ·53 of the radius. Thus I estimate ·20″

[1] This is in agreement with Bosanquet's observation, referred to in para. 3 below, that the correction for the open end of a short tube of a given bore is greater than for a long tube.

as the correction for the open end of my flute. The correction for the mouth end is thus 2·0″ diminished by 0·2″, or 1·8″; hence that for a half inch hole is 2·4 − 1·8=0·6″; and that for a $\frac{3}{8}$″ hole is ·47. This is for a tube of thickness $\frac{1}{8}$″; for a greater thickness the correction would have to be increased. When designing an instrument of another internal diameter, say $\frac{3}{8}$″, the correction for a proportional hole of $\frac{1}{4}$″ will obviously be 0·6″×1/2=0·3″; for a 5/16″ hole it will be 0·24″. It is however convenient to make holes of various sizes; and for holes greater in diameter than ·6 of the bore the correction can be got by extrapolating from those given. But for holes smaller than ·6 of the bore other considerations must be taken into account.

4. The obvious plan for a concert flute with a bore of $\frac{3}{4}$″ would be to make all the holes of the same size, say $\frac{1}{2}$″, not only for the notes rising in semitones from *D* to *C*-sharp of the bottom octave, but also for the bottom *C* and *C*-sharp and also for the auxiliary topmost holes of the *D* and *D*-sharp shake-keys. But if this were done the effect of pulling out the tuning slide to flatten the general pitch would have a much greater effect when the topmost holes were speaking than when the bottom holes were speaking, and the instrument would be out of tune with itself. This defect is greatly diminished by making the topmost holes smaller than the rest; in fact as a hole is made smaller its effect seems to depend more on its size and less on its position. In practice the upper *C*-sharp hole, which is perfectly in tune in the lowest octave, is in a position which would, if the hole were uniform in size with the lower holes, correspond much more nearly to *D* than to *C*-sharp. Also the *D* and *D*-sharp holes, which, like the *C*-sharp, are only about 11/32 in diameter, are in positions closely corresponding to the wave-lengths of *D*-sharp and *E* respectively: thus in the fingering of *B*-flat in the third octave the so-called *D* key is used, as lying nearly in the place of *E*-flat. This same key is used for the fingering of *D*-sharp in the fourth octave, while the *D*-sharp key is opened in the usual fingering for *E* in the fourth octave.

5. To a beginner in the flute playing the most obvious fact in blowing a low note is that as the force of blowing is increased, keeping the lips quite still, the pitch rises slightly, and on reaching a certain amount of force the note will jump up an octave, then becoming usually rather flat in pitch. If the force is further

increased the next harmonic may be reached but it also will be flat in pitch. After some skill has been acquired the player will learn to keep the pitch of a note constant when blowing with varying degrees of force by pushing out his lips when blowing softly and drawing them in and blowing more downwards when blowing with more force. His production of the harmonics will then depend on the position and shape of his lips and each harmonic can be produced either loudly or softly. The control on the pitch of a note that can be exercised by altering the lips is much greater when the topmost hole left open by the fingers is one of those near the mouth-hole than when it is near the bottom end.

6. Let us now consider the question of fingering, or, in other words, the selection of the holes which are to be covered in order to produce the successive notes. For an ordinary concert flute in *D* no question arises over the bottom octave or the middle octave as far as *A* : all that is necessary is to have the 'speaking' hole open and one or two holes below it for 'ventilation,' in order to produce a free tone. When we are wanting the octave above any note, say *B*-flat, we must produce a note which instead of having 'nodes' merely at the mouth-hole and at the *B*-flat hole, has also one half-way between (corrections being ignored). But for such a wave-length there will also be a node at a point whose distance from the mouth-hole is 3/2 times that of the *B*-flat hole. This point corresponds to a note a fifth deeper than *B*-flat. Thus we may produce the note we want by opening merely the *B*-flat hole and the *E*-flat and lower holes: and it is easily verified that while with the ordinary fingering it is difficult to blow the note up to pitch when a very gentle sound is wanted, with this theoretical fingering the gentlest note will not be flat.[1] Naturally with certain patterns of flutes this method is not applicable to all notes. For example, a *B* natural in the middle octave cannot be sounded in this way on an ordinary Boehm flute; for covering the *F* hole will close the hole through which the *B* natural speaks. For the *C* and *C*-sharp in the middle octave we may define the vibrations in the tube still further by a third node,

[1] Fingerings of this type are recommended for pianissimo notes in one of the foreign hand-books.

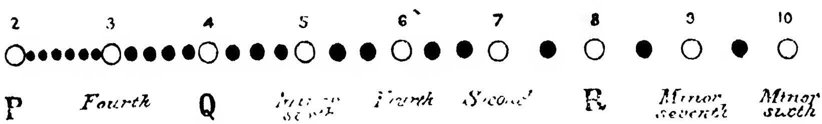
for the loop at 4/2 times the distance from the mouth-hole is the bottom *C* or *C*-sharp key. For the latter note we should have merely three holes open, the upper *C*-sharp hole, the *F*-sharp and the bottom *C*-sharp. In practice however there is no advantage over the theoretical method previously described.

7. For notes in the third octave the methods of cross-fingering to be found in the text-books differ in several cases from the theoretical fingering owing to the mechanical connections between certain of the keys. For the theoretical fingering of any note, say *E*-flat, we remember that the distance from the mouth-hole to the first node is to be a quarter of that from the mouth-hole to that through which the lowest *E*-flat speaks; and we must, as far as possible, open the holes corresponding to all the nodes—that is at 1/4, 2/4, 3/4, 4/4, 5/4, 6/4, 7/4, etc., of this distance. The corresponding frequencies are 4, 2, 4/3, 1, 4/5, 2/3, 4/7, etc. For *E*-flat in the third octave, for example, the ordinary fingering uses the harmonics corresponding to 4/3 and 1; the *E*-flat key is open and that a fourth above this, i.e. the *A*-flat key. There is no key with frequency 2 on the ordinary Boehm flute, for the *E*-flat shake-key is so small that its position is above that really corresponding to *E*-flat for a high note, and if this is opened in addition to 4/3 and 1 the resulting note is too sharp. If however the *D* shake-key is opened instead a good note is obtained. For *F*-sharp the holes to be opened are those producing B, *F*-sharp and *D*; and the note so produced is true if the holes are in their true places. But if the *D*-sharp key is, as usual, kept pressed down for the sake of rapidity of execution, the *F*-sharp note becomes slightly flat; and in order to counteract this error some makers put the *F*-sharp hole in a flute at a trifle higher than its true place. The error in the bottom octaves is very slight and is not so serious as an error in the top octave where the flute is most conspicuous in an orchestra. If the hole for *F*-sharp is not placed above its theoretical position the top *F*-sharp will be rather flat unless the *D*-sharp key is left closed. For *G*-sharp the frequencies 4/3, 1, 4/5, 2/3 correspond to a fourth above, *G*-sharp, a third below and a fifth below: or to *C*-sharp, *G*-sharp, *E*, and bottom *C*-sharp. This fingering gives a ringing note that is a trifle sharp; but it is too complicated for ordinary use, and may be simplified by opening all the holes below the *E*. We may further simplify

by opening the C natural hole near the top of the tube and thus only close the E and F holes in addition to the ordinary fingering on the Boehm flute. This will be useful if the ordinary fingering gives a flat note. Similarly for the high B-flat we open the holes that produce B-flat, G-flat, and E-flat, leaving the holes below this last open. This is rather more complicated than the ordinary fingering but is easily played up to pitch while with the ordinary fingering flatness is hard to avoid, especially when playing softly. The theoretical C-sharp in the topmost octave (with top C-sharp, A, F-sharp and bottom C-sharp keys only open) is a trifle sharp. But if the top C-sharp key be closed the note is in tune. It is however much more complicated than the ordinary fingering, and is of no use except when playing pianissimo.

8. For the few notes of the fourth octave that are of any importance the distances of the holes opened from the mouth-hole

Figure I



must be proportional to 4/8, 5/8, 6/8, 7/8, 8/8, 9/8, 10/8, etc., and the frequencies to 2, 8/5, 4/3, 8/7, 1, 8/9, 4/5, etc. For D these give D, B-flat, G, E (roughly), D, C, and the ordinary fingering for a Boehm flute uses the holes C-sharp (because the D shake-key is too highly placed), B-flat, E and C. The other holes are not available owing to the mechanical relationships between the keys, and perhaps on this account the note is not appreciably sharp. For the E of this octave the holes to open are those of top E, C, A, F-sharp (roughly) E, D, C. The ordinary Boehm fingering uses the D-sharp shake-key, A, F-sharp, E, C, all other holes being closed. In this way the note is difficult to sound; but if the theoretical fingering is followed, the C hole is to be opened in addition and the lower D hole may also be open, i.e. the so-called C-sharp and C-natural keys should not be closed. The sounding of the E then becomes fairly easy; in fact it is easier than the top E-flat or D. If it is found a trifle sharp the D-sharp shake key

Figure II

The keys over the holes for B and G are dotted as in a Boehm flute they are not controlled by the fingers directly, but through other keys.

D$^\sharp$	0	▮	▮	▮	▮	▮	▮	▮	0	▮
D	▮	▮	▮	▮	▮	0	▮	▮	▮	▮
C$^\sharp$	●	●	●	○	○	●	●	○	●	○
C	▮	▮	▮	▮	▮	▮	▮	▮	0	▮
B	⊜	⊜	◌	⊜	⊜	⊜	◌	⊜	⊜	⊜
B$^\flat$	●	○	○	●	●	○	○	●	●	●
A	○	●	●	●	○	●	○	○	○	●
G$^\sharp$	▮	▮	▮	0	▮	▮	▮	▮	▮	0
G	⊜	⊜	⊜	⊜	⊜	⊜	⊜	⊜	⊜	⊜
F$^\sharp$	●	●	○	●	●	○	○	○	○	○
F	●	○	●	●	○	●	○	●	●	●
E	○	●	●	○	●	●	●	●	○	○
D$^\sharp$	▮	▮	▮	▮	▮	0	0	0	▮	▮
D	●	●	○	●	○	○	○	●	○	○
C$^\sharp$	●	○	○	○	○	○	○	○	○	○
Note	E	F	F$^\sharp$	G$^\sharp$	A	B$^\flat$ 1	B$^\flat$ 2	C$^\sharp$	E	F$^\sharp$

The fingerings of E and F are solely of theoretical interest. For F$^\sharp$ and A closing the D$^\sharp$ hole keeps the note from being flat if there is any tendency in that direction. The G$^\sharp$ fingering may be simplified by opening the lower D and D$^\sharp$ holes. For B$^\flat$ the first fingering is strictly theoretical and is inclined to be sharp. So it is better to close the upper D key as in the second fingering. For C$^\sharp$ there is no difficulty in closing the D key while opening the D$^\sharp$, but either of these alone will suffice.

may be closed. In the same way the theoretical *F*-sharp, with the *C*-sharp, *G*-sharp, *F*-sharp, *E*, *D* holes open (not the *B* since opening this involves opening the *B*-flat also) may be sounded; but it is too difficult to be of practical value.

9. Some of these fingerings are illustrated in the diagrams which are appended. Figure 1 is a general scheme for the theoretical fingering of any note above the first octave. In it are indicated a series of holes a semitone apart of which the open circles represent open holes and the black circles closed: the holes extend over a greater range than an ordinary flute contains, but only those actually on the flute for the note in question need be considered. The numbers immediately above the line are proportional to the distances of the holes in question from the mouth-hole; so that from 2 to 4 is an octave and from 4 to 8 is an octave. Let us now suppose that we want the note which is an octave above that corresponding to *P*, or two octaves above that corresponding to *Q*, or three octaves above *R*. The open circles will then obviously indicate the holes to be opened, and their intervals above the fundamental sounded note will be those given immediately below. If, for example, we want the top *B* which is two octaves above the *B* in the lowest octave, the hole marked 4 must be the *B* hole; the fourth above is *E*, so the *E* hole above this is to be opened if there be one. The hole marked 5 is a minor sixth above *B*, or *G*; the hole marked 6 is a fourth above or *E*; that marked 7 is a second or *C* sharp; that marked 8 is *B* which is beyond the range of the ordinary flute. On a Boehm flute the hole for 4 is the *D*-sharp shake-key, while the closing of the holes between 5 and 6 will close the hole for *G*, so all holes below the *G* hole are left open: the note produced is sufficiently controlled by the five holes from 4 to 5, and the opening of all holes below the *G* practically cancels the lower part of the tube.

10. In Figure 2 are given some fingerings of high notes on a Boehm flute, indicated by theory where these differ from the standard to be found in the ordinary text-books. The notes in question are *E*, *F*, *F*-sharp, *G*-sharp, *A*, *B*-flat and *C*-sharp in the third octave, with *E* and *F* sharp in the fourth octave.

X. On Wave-Propagation in Optically Heterogeneous Media, and the Phenomena observed in Christiansen's Experiment.

By Nihal Karan Sethi, M.Sc., Assistant Professor of Physics in the Benares Hindu University.

(Plate VII.)

CONTENTS.

I.—Introduction.

The very striking and instructive optical experiment known under his name was first described by Christiansen in 1884.[1] He found that glass or other isotropic transparent solid in the state of fine powder which is ordinarily white and quite opaque, becomes transparent in respect of a limited region of the spectrum on immersion in a mixture of carbon disulphide and benzol in proportions suitable to make its refractive index nearly equal to that of the powder. The mixture exhibits a beautiful display of colour in the transmitted light and in the halo with which bright objects appear to be surrounded when viewed through it. A general description of the effects observed will be found in Prof. R. W. Wood's Treatise on Physical Optics.[2] It would appear,

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[1] Annalen der Physik XXIII, p. 298.
[2] Second edition, p. 110.

however, that in spite of the very attractive features of the experiment, the phenomena exhibited by these mixtures have not yet been thoroughly investigated. In two important papers, the late Lord Rayleigh[1] drew attention to Christiansen's experiment, especially to one remarkable feature observed in it, namely, the very small range of wave-lengths transmitted by these mixtures, and showed how this feature might reasonably be explained on the principles of the wave theory. Apart from this, however, there are a number of other important questions which arise and as yet remain to be answered. What is the exact distribution of intensity of light among the various wave-lengths transmitted? How does this distribution depend on the size of the particles in the powder and the length of path of light through the mixture? Has the halo a definite structure, or must we be content with vaguely describing it as having a colour complementary to that of the transmitted light? How is the halo influenced by the factors mentioned above and the other circumstances of the experiment? It is proposed in the present paper to examine these and other questions relating to the propagation of light in optically heterogeneous mixtures. It is believed that the inquiry may be of interest, especially in view of the very wide application in recent years of the method of immersion in the determination of refractive indices and dispersive powers.

II.—THE TRANSMITTED LIGHT.

According to the principles of geometrical optics, a mixture of two isotropic media should regularly transmit only the rays for which the refractive indices of its components are identical, and should scatter all others in various directions. Actually however, in Christiansen's experiment, we find that this is not the case and that part of the incident light is regularly transmitted and is capable of giving well-defined optical images even when the refractive indices differ by an appreciable quantity. For instance, if a flat-sided cell containing the mixture be placed between the collimator and the prism of a spectroscope of which the slit is illuminated by sunlight, we find that a finite region of the spectrum continues to be visible and is in sharp focus, the Fraunhofer lines

[1] Scientific Papers, Vol. II, p. 433 and Vol. IV, p. 392.

being quite clearly distinguishable in every part of it. The range of wave-lengths thus regularly transmitted in the spectrum varies with the fineness of the particles and the thickness of the layer of powder. Christiansen in particular found that the transmitted region increased in width with the fineness of the powder used and indeed comprised nearly the whole visible spectrum when the observations were made with the finest powders of all separated by prolonged elutriation. The explanation of these effects is obviously of considerable interest.

The late Lord Rayleigh[1] proposed a treatment on the following lines. He first determined the probability that the number of particles of one component which a ray will encounter during its passage through the mixture, will differ from the mean number m by less than $\pm r$, and showed that this will be sufficiently great and equal to ·84 for a value of $r=\sqrt{2m}$. This gives a measure of the phase differences likely to arise in the passage of the light through the heterogeneous medium, from which it is inferred that the range of wave-lengths transmitted may be expected to be $\frac{m}{\sqrt{2m}}$ times that which is not resolved by a prism of equal thickness with a dispersive power equal to the difference between the dispersions of the two media. The expression for the latter resolving power is well known to be

$$\delta\lambda = \frac{\lambda}{t\,\dfrac{d\mu}{d\lambda}}$$

and even if m is taken to be equal to $\frac{t}{d}$ —a value evidently greater than the real one—the total range of wave-lengths transmitted by the powder should be

$$\Delta\lambda = \sqrt{\frac{t}{2d}} \cdot \frac{\lambda}{t\,\dfrac{d\mu}{d\lambda}}$$

$$= \frac{1}{\sqrt{2td}} \cdot \frac{\lambda}{\dfrac{d\mu}{d\lambda}} \qquad \text{(1)}$$

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[1] Scientific Papers, Vol. IV, p. 395.

where t is the thickness of the layer of the medium and d is the average size of the particles in it.

It would appear, however, from the investigation made by the present writer that the observable width of the region of transmission in the spectrum depends to a large extent on the intensity of light used and in all the measurements made, it was found to be much greater than that indicated by Rayleigh's formula. Even Christiansen's estimate is much greater than what equation (1) will give for his data. The fact is that the distribution of intensity among the wave-lengths transmitted is of the exponential form, being maximum for $\mu-\mu'=0$ and diminishing more or less rapidly to zero for large values of $\mu-\mu,'$ (μ and μ' are the refractive indices of the two components of the mixture). We cannot, therefore, strictly speaking, talk of a region of transmission without at the same time defining the limiting intensity of the light at the ends of that region.

Fortunately, considerations of the wave-theory of light combined with the theory of probability readily give us an expression for the intensity of the transmitted light. We may treat this problem on much the same lines as adopted in a recent paper by Mr. Chinmayanandam[1] in discussing the specular reflection from a rough surface. Consider the configuration of the wave-front immediately on emergence from the heterogeneous medium. It will not be a perfectly plane wave-front, because different portions of it will have been unequally retarded by the random distribution of the particles of the powder. We may assume that the deviations of the wave-front from perfect planeness follow the well-known law of errors. Certain parts of the wave-front would have traversed more of the solid and less of the liquid and *vice versa*. Let x denote the difference between the average thickness of the solid traversed by the whole wave-front and the thickness traversed by a specified element of the wave-front. Then the total area of the portions of the wave-front in advance of or behind the mean, for which this difference lies between x and $x+dx$, may be taken proportional to $e^{-Ax^2}dx$. The resultant vibration due to these portions of the wave-front is, therefore,

[1] Physical Review, Vol. XIII, Feb. 1919, p. 96.

$$y = e^{-Ax^2} \cos \left\{ wt - \frac{2\pi}{\lambda} (\mu - \mu')x \right\} dx ,$$

if the equation of the vibration for the mean wave-front be $y = \cos wt$. The average vibration due to a unit area of the complete wave-front will then be for the wave-length λ,

$$y = \frac{\int_{-\infty}^{+\infty} e^{-Ax^2} \cos \left\{ wt - \frac{2\pi}{\lambda} (\mu - \mu')x \right\} dx}{\int_{-\infty}^{+\infty} e^{-Ax^2} dx} .$$

This will make the average intensity

$$I = e^{-\frac{2\pi^2}{A}\left(\frac{\mu - \mu'}{\lambda}\right)^2} = e^{-B\left(\frac{\mu - \mu'}{\lambda}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

where the constant B will depend on the size of the particles d and the thickness of the layer of powder t and possibly to some extent on the arrangement of the particles.

To test the foregoing formula and to find experimentally the manner in which the constant B depends on d and t, a number of observations were made by the writer. Powdered glass was the substance used being the most convenient and suitable for the purpose. By passing it in succession through several sieves of wire gauze and cloth having a different number of meshes to the inch, powders of varying grades of fineness were obtained, the particles in each of which lay between definite and determinable limits of size. The samples of powder thus obtained were then thoroughly cleaned, washed and dried. For the purpose of the observations, the powder was placed in a cell with parallel faces of optically good glass, and enough carbon disulphide was poured into the vessel. Benzene was then slowly added till the refractive index μ of the glass and μ' of the liquid mixture became equal for red light and this began to be freely transmitted. Further additions of benzene shifted the region of transmission towards the violet end of the spectrum, thus affording opportunity for making observations at any desired stage. The thickness t of the layer through which the light had to pass was varied by inserting into the powder pieces of clean glass plate of different thicknesses.

The constant B was determined by placing the cell containing the powder between the collimator and the prism of a wave-length spectroscope and determining the range of wave-lengths transmitted, then screening the light by a screen which transmitted a known fraction k of the light falling on it, and then again determining the diminished range of transmission. The source of light must be quite steady and a 500 c.p. half-watt lamp was used. It is clear that if the limit to which the eye of a particular observer in particular circumstances can just see, corresponds to a certain absolute value of intensity, the value of the intensity at the limit in the second case after screening must be equal to that at the limit in the first case. So that

$$ke^{-B\left(\frac{\mu-\mu'}{\lambda}\right)_2^2} = e^{-B\left(\frac{\mu-\mu'}{\lambda}\right)_1^2}$$

and

$$B = -\frac{\log_e k}{\left(\frac{\mu-\mu'}{\lambda}\right)_1^2 - \left(\frac{\mu-\mu'}{\lambda}\right)_2^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

For comparison three such screens were employed for which the values of k as determined by a rotating sector photometer were $\frac{1}{8{\cdot}41}$, $\frac{1}{13{\cdot}44}$ and $\frac{1}{22.5}$ respectively, and the values of B determined by each of them separately agreed well. The variation of B with t and d is set forth in tables I and II.

TABLE I.

$d = {\cdot}0055$ c.m.

t in c.m	B	$\frac{B}{t}$
1·1	·0528	·0480
·78	·0315	·0404
·45	·0161	·0358
·30	·0130	·0433
·12	·00705	·0588

TABLE II.

$t = \cdot 78$ c.m.

d in c.m.	B	$\frac{B}{d}$
·0055	·0315	5·69
·0142	·0515	3·63
·0245	·0900	3·67
·0410	·206	5·03

In view of the unavoidable uncertainty in the observations and changes in the temperature of the mixture and also the non-constant nature of the arrangement of the particles, the values of $\frac{B}{t}$ and $\frac{B}{d}$ may be seen to be fairly constant for the large variations of the values of t and d. So that B must be taken to be of the form

$$B = ctd.$$

and
$$I = e^{-ctd\left(\frac{\mu-\mu'}{\lambda}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

where c is a constant which may depend on the arrangement of the particles but is otherwise an absolute one. It may be noted that this formula makes the dimensions of c as of a pure number.

The result $B = ctd$ may also be deduced theoretically if we combine (1) and (2). If we remember that

$$\tfrac{1}{2}\Delta\lambda \frac{d\mu}{d\lambda} = (\mu - \mu')$$

at the limit of visibility, we get

$$\frac{\mu-\mu'}{\lambda} = \frac{1}{2}\frac{\Delta\lambda}{\lambda} \cdot \frac{d\mu}{d\lambda} = \frac{1}{\sqrt{8td}}$$

$$\therefore \left(\frac{\mu-\mu'}{\lambda}\right)^2 = \frac{1}{8td} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Now, if we suppose that at the limit given by Rayleigh's formula, the intensity falls down to a fixed value, we must have from (2)

$$B\left(\frac{\mu-\mu'}{\lambda}\right)^2=\text{constant}$$

from which (5) will give

$$\frac{B}{8td}=\text{constant}$$

or

$$B=ctd.$$

The average value of the absolute constant c as deduced from the experiments described above is about 7.

From the expressions given above, it is a simple matter to calculate the proportion of the energy that appears in the transmitted light, given the size of the particles, the thickness of the layer and the dispersive powers of the media or *vice versa.* It would be interesting to continue the work and test experimentally whether the formula given by the foregoing theory continues to be valid in the case of powders much finer than those used by the author in his observations and which transmit a much larger portion of the spectrum. No quantitative estimate is given by Christiansen of the size of the particles in the case of the finest powders used in his observations which according to him transmit nearly the whole of the visible spectrum. Otherwise this question could very easily have been tested. It seems not unlikely that the opalescent coloured precipitates of potassium fluosilicate experimented upon by Wood may be usefully employed for further work on this point.

III.—The Colours of the Halo.

The energy which fails to appear in the transmitted light goes mostly into the halo seen surrounding the light-source, the proportion lost by reflection or absorption being small except possibly in the case of large thicknesses. Some observations have been made by the writer of the manner in which the scattered light is distributed in the halo. A strong source of light— an arc-lamp or a 1000 c.p. half watt lamp—was placed behind a small hole which was observed through the mixture of the powder and the liquids from a distance of about two meters. Naturally, the source appeared to have the colour of the transmitted light and was surrounded by a halo which was also coloured and which extended to very large angles. Hitherto the halo has been described as

having a colour complementary to that of the transmitted light. This suggests that the halo is all of one colour. But the most casual observation was enough to show that this was not the case and that the colour of the halo changes progressively from the centre outwards. The part of the halo immediately surrounding the source is of nearly the same tint as the source itself and shows a fine mottled or granular structure similar to that exhibited by diffraction halos in monochromatic light which has been discussed by De Haas[1] in a recent paper. The following table shows the colours of the different parts of the halo in various cases, the order being from the centre outwards:—

TABLE III.

TRANSMISSION.	HALO.
Red	Red, Orange, Yellow, Greenish-yellow, Green, Blue, Indigo.
Yellow	Yellow, Orange, Orange-yellow, Yellow, Yellowish-green, Green, Greenish-blue, Blue, Indigo.
Green	Green, Orange-yellow, Rose, Lilac.
Blue	Blue, Greenish-yellow, Orange-yellow, Rose, Red.
Indigo	Indigo, Blue, Bluish-white, Greenish-white, Yellowish-white, Reddish-white.
Violet	Violet, Blue, Bluish-white, Greenish-white, Yellowish-white, Reddish-white.

The explanation of the variation of the colour of the halo in its different parts becomes evident on examining the spectrum of the light scattered by the powder. In order to do this, all that is necessary is to take a direct-vision spectroscope and point it towards the cell containing the mixture. But the light proceeding in directions other than the one desired must be prevented from falling on the slit of the spectroscope. This can be ensured if the distance between the powder and the spectroscope slit is sufficiently

[1] K. Akad. Amsterdam Proc. XX, pp. 1278–1288, 1918.

great (over a meter or two) and the area of the surface of the powder sending out the light to the slit is itself considerably restricted by a narrow aperture. Specially interesting are the effects observed at very small angles. Fig. 1 in the plate shows the spectrum in the direction of the freely transmitted light and consists of a very narrow band for which $\mu-\mu'$ does not appreciably differ from zero. Figs. 2–5 in the plate are the spectra at gradually increasing obliquities. Fig. 6 is the complete spectrum reproduced for comparison of the widths of the bands obtained in the various cases. From these it is clear that proceeding from the centre, the halo consists at first mainly of two very narrow bands in the spectrum on either side of the region of transmission; these bands gradually widen out and also become more and more separated as the obliquity of observation is increased. The actual colour of the halo in any case is the resultant due to the mixture of the two bands in the spectrum. At a sufficiently great obliquity, these bands will together comprise almost the whole of the spectrum except for the wide region included between them in which the intensity is deficient or actually zero. In this case the colour of the halo may be said to be roughly the complementary of that of the transmitted light—roughly because a considerable part of the spectrum is missing from it over and above the portion regularly transmitted through the powder.

Further information regarding the halo is obtained by using a monochromatic light-source—say the green light of the mercury-vapour lamp—and varying $\mu-\mu'$ for this light by altering the proportions of carbon disulphide and benzene in the mixture. But a better and more convenient method is to keep the mixture unaltered and change the wave-length of the light used. A monochromator must be used for this purpose. It is most instructive to observe, as the screw of the instrument is slowly turned, the halo decrease in size, vanish altogether—the intensity of the regularly transmitted light being at this instant a maximum—and then increase once again. From the general appearance of the monochromatic halo and some photometric observations made by the author, it would appear that the distribution of light in it is at least approximately of the form e^{-ax^2}, the illumination being maximum at the centre and gradually diminishing symmetrically to zero. The constant a must also decrease with increasing $\mu-\mu'$ and

N. K SETHI PLATE VII.

SPECTRA

Fig 1

Transmission

Fig. 2

Scattering

Fig 3

Ditto
at greater obliquity

Fig. 4

" "

Fig. 5

" "

Fig. 6

Complete Spectrum

Fig 8

Scattering by
Glass

Fig. 9

Ditto
Quartz

Phenomena observed in Christiansen's Experiment.

very rapidly too; for the extent of the halo, as far as the eye can see it, is very small at first and increases very rapidly as $\mu-\mu'$ increases. It is also clear that it should be an even function of $\mu-\mu'$ so that at least as a first approximation a may be taken to be of the from $\frac{b}{(\mu-\mu')^2}$, where b depends on the size of the particles d, the thickness of the layer of powder t and possibly on the wave-length λ.

In fig. 7 are drawn a few curves of the form $y=e^{-ax^2}$ for different values of a and correspond to the illumination curves for different wave-lengths. The abscissae represent the distances from

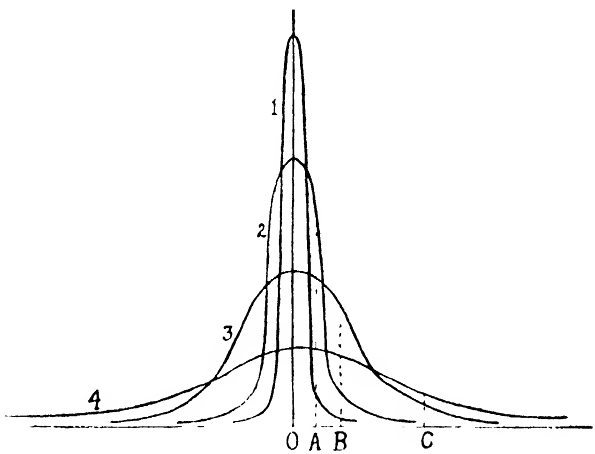

FIG. 7.

the centre of the halo. Curves 1, 2, 3, 4, correspond to cases in which the difference $\mu-\mu'$ has steadily increasing values. It is at once apparent from the figure that at the point A of the halo, λ_1 corresponding to curve 1 is practically absent and the principal illumination is due to wave-lengths near about λ_2 and λ_2' (one greater and the other smaller than λ_1) corresponding to curve 2. Similarly at B, even λ_2 and λ_2' are almost absent and the main illumination is due to wave-lengths in the neighbourhood of λ_3 and λ_3'. This completely agrees with the photographs in figs. 2–5 and satisfactorily explains why the dark band in the centre and the two bright bands on either side gradually widen out as we proceed further and further out from the centre of the halo.

The writer is not yet in a position to elucidate the exact manner in which the numerical value of the constant b determining the angular size of the halo depends on d, t and λ, or to offer a detailed theory. The general nature of the variations was, however, evident from the observations, namely that the halo increases in size when the particles in the mixture are smaller or the thickness of the layer is larger. Much more photometric work is necessary before this question and the manner in which λ comes in can be definitely settled. Reference may, however, be made here to the explanation of the granular structure of the central part of the halo which has already been mentioned. This is evidently connected with the arbitrary variations in the phase at different points of the wave-front emerging from the mixture. Any quantitative theory of the distribution of light in the halo can give only the statistical or average value of the intensity in the neighbourhood of any given direction. At individual points of the halo, large deviations from this average value are inevitable. Near the centre of the halo, as we have already seen, the scattered light is confined to a very small region of the spectrum, and hence the arbitrary fluctuations of intensity are specially marked. Further out from the centre of the halo, the effects of different parts are superposed and the irregular fluctuations of intensity are less easily noticeable.

That the size of the halo surrounding the source varies with the difference $\mu - \mu'$ of the refractive indices and hence is different for different colours of the spectrum is very clearly illustrated by fig. 8 in the plate. This is a picture obtained by interposing the cell containing a uniform layer of the mixture in the path of the rays forming a focussed spectrum on the photographic plate, the distance between the plate and the cell being about half a centimeter. Owing to the scattering action of the layer, the different parts of the spectrum are spread out to different extents, this spreading being evanescent for the point at which $\mu = \mu'$. There is a distinct asymmetry in the photograph which would seem to indicate that the size of the halo is greater on the large wave-length side and less on the other.

IV.—THE SURFACE COLOURS.

A very striking effect which does not appear to have been

previously described is the brilliant colour of the boundary between the glass powder in the lower part of the vessel and the liquid above it. This colour is different from the rest of the powder and is ever so much more brilliant. In order to observe it well, it is necessary that the surface of the powder should be level, which can be easily secured by gently tapping the vessel, and the line of sight should be almost tangential to this surface. Another very remarkable thing about this boundary is that it does not show the same colour on both sides, so that when regarded from the lower or the glass side, it shines with one colour which changes into quite a different one when the observation is made from the upper or the liquid side. Both these appearances are to some extent summarised for the various stages in the following table in which the order of the colours is for gradually increasing angles with respect to the transmitted light, the most beautiful stage being when the yellow is freely transmitted.

TABLE IV.

TRANSMISSION.	LIQUID SIDE.	POWDER SIDE.
Red.	Green, Blue.	Deep Red.
Yellow.	Green, Blue, Violet.	Orange, Red.
Green.	Blue, Violet.	Yellow, Orange, Red.
Blue.	Violet.	Green, Yellow.
Violet.	Faint Violet.	Green, Yellow.

The reason for these colours appears to be that the surface of separation acts more or less like a totally reflecting barrier, those wave-lengths for which $\mu > \mu'$ being totally reflected on the glass side, while those for which $\mu < \mu'$ appear on the liquid side.

The fact that the boundary colours appear brilliant when the line of sight is almost tangential to the surface is easily explained if we remember that the difference of refractive indices is at most very small and hence the angle of incidence should not differ much from a right angle.

These colour effects are identical with coarse as well as fine powders, the only difference being that with fine powders the boundary line appears much sharper and more well defined.

V.—Experiments with Liquids.

The colours obtained when equal volumes of glycerine and turpentine are shaken together have been noticed previously and Wood remarks that they are of a similar nature to those observed in Christiansen's experiment. It was thought that it would be worth while to examine this case more closely. At room temperatures, pure glycerine has a refractive index slightly less than that of spirits of turpentine throughout the visible spectrum, the difference being greatest at the violet end and least at the red. The effects noticed at ordinary temperatures are not therefore particularly striking. The author has found however that at higher temperatures very lively colours may be obtained. The liquids may be put together in a small flat-sided flask which is corked and then warmed up by being put in a beaker containing boiling water for a few minutes. On taking it out and shaking up the mixture and looking through it at a source of light, we find no transmission at first but merely a halo with violet centre surrounded by the other spectrum colours in regular order. As the mixture cools, violet, blue, green, yellow and red light is transmitted in succession and the colour of the halo undergoes corresponding changes. Finally, the transmitted light vanishes, and we have only a halo with a yellowish white centre and a blue-violet margin. The size of the halo in all these cases is much smaller than in the case of the glass powder and consequently it is much easier to distinguish the colour of the different parts of the halo. Another interesting point about this halo is that the mere shaking up of the mixture increases its size as one should naturally expect from the greatly diminished size of the drops.

The upper liquid, viz. turpentine, becomes very quickly free of the large drops of glycerine which fall down rather rapidly, but very fine drops remain floating in it for a considerable time. When an object is viewed through this apparently clear liquid, a halo is seen to surround it, while the object itself remains practically white. This halo is quite different from the halo referred to above and which is seen through the lower liquid where one can see many

large drops. It is much smaller in size and does not show much gradation of colour. Generally only one or two tints are visible. When the transmission is near the red end of the spectrum, this halo consists of blue and violet colours only. When the transmission is towards the other end, the halo is brown and red. In the intermediate stages it has a purple or a pink colour. Both this and the proper halo show a very curious appearance of activity near their centres which strongly reminds one of the appearance in a spinthariscope. This is no doubt due to the drops slowly falling down through the liquid.

Looking at the surrounding objects in daylight through the hot mixture, the general transparency is found to be very great, the range of spectral transmission being rather considerable. Observed with a direct vision spectroscope, the transmitted light appears not as a sharp band as in fig. 1 in the plate but much wider. Similarly a spectrum seen through this mixture when hot shows a much broader region in good focus than does the glass powder. This is no doubt due to the smaller difference in the dispersive powers of the constituents of the mixture. The same is true of the halo also. The dark bands corresponding to figs. 2–5 are obtained but they are never so narrow.

As the small drops fall through the liquids, something like a boundary of separation forms between the clear turpentine above and the collection of drops below. This boundary like the one in the case of the glass powder shows brilliant colours different from the rest of the mixture and also different on the two sides. When, however, the drops become large, there is nothing like a regular boundary and no such effects are produced. Individual drops no doubt produce certain diffraction effects and their edges shine out with various colours depending on the difference of refractive indices of the two liquids and the angle of observation. These peculiar effects are visible not only in the drops just below the clear turpentine, but also in practically all other drops. This gives rise to a sparkling appearance in the liquid and is particularly noticeable when the mixture is just taken out of boiling water and its temperature is rather high. These effects will be discussed in a separate paper.

It may be worth while also to record the results observed when glycerine is shaken up with rather a small quantity of

turpentine. In this case, a milky emulsion of the two liquids results, which takes a very long time to clear and which may, therefore, be very conveniently used for observations of the effect of warming on its transparency. Partly on account of the very fine state of subdivision in the mixture thus obtained and partly no doubt also on account of the small quantity of turpentine taken up by the glycerine, the region in the spectrum transmitted by it when warmed up is very great in this case.

VI.—THE COLOURS OF DOUBLY REFRACTING POWDERS.

Christiansen mentions in his paper that doubly refracting powders do not give satisfactory results in his experiment, as the colours they show are not so pure, and the mixture is never fully transparent. On the other hand, we find that Prof. R. W. Wood remarks that the best results that he could obtain were with powdered quartz. It is difficult to reconcile these contradictory statements, for although it may be true that so far as the general appearance of the powder and the halo is concerned, powdered quartz does perhaps give more attractive and lively colours than does glass and the halo is also much wider, yet there does not remain the slightest doubt after a closer examination of the phenomena that Christiansen was absolutely correct in his statement referred to above. We find in fact that in the case of quartz the whole of the light that emerges through the powder appears in the halo and there is, strictly speaking, no light regularly transmitted and capable of forming a definite image of the source. The real Christiansen phenomenon is as a matter of fact altogether absent. This is apparent on observing a small brilliant source—the filament of an electric lamp or a narrow aperture backed by an arc lamp—through a layer of the mixture containing the quartz powder, when it will be found that even with very small thicknesses used, there is no defined image of the source. It remains invisible and we merely see a halo surrounding its position the colours of which gradually alter as we proceed from the centre outwards. It would therefore seem that what Prof. Wood calls the "transmitted colour" was merely the central portion of the halo and he could not have seen the actual image of the lamp flame.

The spectroscopic examination of the phenomenon in the manner described in the preceding sections makes this all the

more clear. None of the appearances shown in figs. 1–5 in the plate can be observed with quartz powder. Even in the direction of the incident light almost the complete spectrum is visible with those portions of it somewhat more intense for which the refractive index of the liquid approaches that of quartz. In directions more and more oblique to the incident light, this brighter portion gradually loses, while the portions on either side of it gain in intensity, but the difference between them becomes appreciable only at rather large angles. The colours observed either in the direction of the incident light or far out in the halo are therefore all impure by the admixture of a fair quantity of white light.

When the cell containing the mixture of powdered quartz was placed between the collimator and the prism of the spectroscope, no portion of the spectrum could be seen. There was only a general illumination of the field.

The complete absence of the transmitted light is, however, best illustrated by fig. 9 in the plate which is the ordinary spectrum as seen through the quartz powder. Fig. 8 is the spectrum seen through the glass powder under similar circumstances It will be noticed that while in fig. 8 there is a certain region of the spectrum which appears to be well focussed and free from halo, there is no such region in fig. 9. The halo does no doubt attain a minimum size, but it does not vanish at any place. And this happens even for very small thicknesses of the powder.

This result may appear to be surprising at first sight, but is easily explained. For, on account of the doubly refracting nature of the quartz, and the absolutely random orientation of the particles with respect to their crystallographic axes, no ray can travel through the mixture from start to finish with one definite velocity. Whenever it will encounter a particle of quartz, it will in general be broken up into two, travelling with different velocities and in some cases when although it is not broken up into two, an ordinary ray may be wholly transmitted as an extraordinary ray. So that every ray must be regarded as partly ordinary and partly extraordinary during its passage through the powder. No possible value of the refractive index of the liquid mixture can allow such a ray to pass as through a homogeneous medium.

The question then arises, "To what value of μ' does the minimum size of the halo in fig. 9 correspond?" In order to settle

this, a wave-length spectroscope with a shutter eye-piece was used. The shutter was closed down to a narrow slit so that the light passing through it had a definitely known wave-length which could be quickly altered. With such an arrangement the wave-length corresponding to the minimum halo could be quite accurately determined.

A suitable refractometer not being available, the refractive index of the liquid corresponding to the minimum halo was found out by a rather indirect but quite a simple method. A tiny little prism of the same quartz (one of the broken pieces) was suspended in the liquid above the powder. Looking through this, one could see two images of the slit in the shutter eye-piece. It was easy to find which of them corresponded to the ordinary ray, for in the first place μ_0 is smaller than μ_e, and secondly the ordinary image did not move as the prism was turned, while the other moved so that the distance between them altered. By turning the wave-length screw, it was easy to make the ordinary image coincide with the slit as seen directly from above and below the prism. For this value of λ, $\mu'=\mu_0$. From the known values of the refractive indices of carbon disulphide and benzol for this wave-length the proportion of the two liquids in the mixture was calculated[1] so as to satisfy the condition $\mu'=\mu_0$. The proportion having been found, the μ' for the wave-length corresponding to the minimum halo could be calculated. A specimen result is quoted below :—

minimum halo at	$\lambda_1=5270$
$\mu'=\mu_0$ at	$\lambda_2=5490.$
But	$\mu=1{\cdot}5454$ at λ_2
$\therefore$	$\mu'=1{\cdot}5454$ at λ_2
and	$=1{\cdot}5488$ at λ_1.

Theoretically one would expect on an average half the path of a ray through the powder to be ordinary and half extraordinary. So that if μ' is equal to the mean of the ordinary and the mean extraordinary indices, or in other words if

$$\mu'=\tfrac{1}{2}\left\{\mu_0+\tfrac{1}{2}(\mu_0+\mu_e)\right\},$$

[1] See Preston: Theory of Light, page 134.

the path difference between a ray passing through the liquid and one through the quartz may be expected to be the least. This value of μ' should, therefore, give rise to the minimum halo. Now, for

$$\lambda=5270,\ \mu_0=1{\cdot}5465 \text{ and } \mu_e=1{\cdot}5551$$

$$\therefore \qquad \mu'=\tfrac{1}{2}\left\{1{\cdot}5465+\tfrac{1}{2}\,(1{\cdot}5465+1{\cdot}5551)\right\}=1{\cdot}54865$$

in complete agreement with the value obtained experimentally.

VII.—Summary and Conclusion.

The paper describes the results of a detailed study, both experimental and theoretical which has been made of the phenomena observed in Christiansen's experiment in which the powder of a transparent substance is immersed in a mixture of carbon disulphide and benzol having a refractive index nearly equal to that of the powder.

(1) *Transmitted Light.*

(*a*) The observable range of wave-lengths transmitted by the mixture has been found to depend on the intensity of the incident light.

(*b*) A theoretical treatment based on the principles of the wave-theory and the theory of probability has been given, and it shows that the intensity of the transmitted light is given by the expression

$$I=e^{-ctd\left(\frac{\mu-\mu'}{\lambda}\right)^2}$$

The influence of the size of the particles and the thickness of the medium on the intensity of the transmitted light has been confirmed experimentally and the value of the absolute constant c has been determined and found to be roughly equal to about 7.

(*c*) The formula shows that with the very finest powders, the range of transmission may be very considerable.

(2) *The Colours of the Halo.*

The statement made hitherto that the colour of the halo is complementary of that of the transmitted light is not correct. The

halo is not of one colour throughout, but has a definite structure. This is shown by observations and photographs of the spectrum of the halo, which at small obliquities consists of two narrow bright bands separated by a dark interval. These bands widen out as the angle of observation is increased. The facts are explained by observations in monochromatic light which indicate that the distribution of intensity in the halo may be taken to be of the form

$$e^{-\frac{b}{(\mu-\mu')^2}\cdot r^2}$$

which is of the exponential type, where b increases as the size of the particles is increased and as the thickness of the layer or the wave-length of light is diminished.

(3) *The Surface Colours.*

The level surface of separation between the clear liquid on the top and the powder below exhibits remarkably brilliant colours which are not only different at different stages of the mixture as regards the refractive indices, but are also different on the two sides of the boundary. This effect does not appear to have been noticed so far. Presumably it is due to a sort of total reflection from the boundary.

(4) *Liquid Mixtures.*

A mixture of glycerine and turpentine shows similar phenomena, but these become evident only when the flask containing the mixture is heated by immersion in hot water. At ordinary temperatures there is no light transmitted. The halo in this case is, however, much smaller, but the general transparency of the mixture is very great. Some interesting diffraction effects by the edges of the liquid drops are also observed.

(5) *The Colours of Doubly Refracting Powders.*

Doubly refracting powders, such as quartz, are, contrary to the statement of Prof. R. W. Wood, unsuitable for the exhibition of the true Christiansen phenomenon, because in their case there is no regularly transmitted light. The whole of the light emerging through the powder appears in the halo, which is differently coloured in its different parts. There is a particular wave-length

for which the size of the halo is a minimum and this has been shown both experimentally and theoretically to correspond to the case in which the refractive index of the liquid is equal to

$$\frac{1}{2}\left\{\mu_0+\frac{1}{2}(\mu_0+\mu_e)\right\}.$$

In conclusion the writer has much pleasure in recording his indebtedness to Prof. C. V. Raman who took considerable interest throughout the progress of the work and whose suggestions and criticisms were of immense help. The experimental work was carried out at the Laboratory of the Indian Association for the Cutlivation of Science.

Dated Calcutta,
The 2nd of November 1920.

XI. On the Production of Musical Sounds from Heated Metals.

By B. N. Chuckerbutti, M.Sc., Assistant Professor of Physics, Calcutta University.

(Plates VIII to X.)

CONTENTS.

I.—Introduction.

That vibrations giving rise to musical sounds may be produced under suitable conditions by the contact of metals at different temperatures, has long been known as a fact of observation and is best illustrated by the familiar piece of apparatus known as Trevelyan's Rocker. As is well known, this apparatus usually consists of a bar of metal (generally brass) prismatic in form which has a narrow longitudinal groove cut along its lower edge, so that if it be placed on a table, the metal rests on two adjacent parallel ridges. If displaced from its position of equilibrium, the bar rocks to and fro, resting alternately on the two lines of support, oscillating at first slowly and then more quickly as the amplitude dies away, the energy being gradually dissipated in the succession of impacts of the points of support upon the table. In actual use the rocker has a handle which consists of a brass rod with a ball at one end. The rocker is heated and is then supported in a slanting position with the edges of the groove touching the clean cold surface of a horizontal block of lead, the end of the handle resting on the

table. The rocker has generally to be started, that is, one edge has to be lifted out of contact with the lead block and then allowed to come down. The movement then continues to sustain itself, a rough sound being heard at first which later gives place to a tone of markedly musical character.

The generally accepted explanation of the working of the Trevelyan rocker (due in the first instance to Sir John Leslie) was developed by Faraday.[1] The correctness of Faraday's explanation was questioned by Forbes,[2] but was supported by Seebeck[3] and later by Tyndall,[4] who carried out an extensive series of studies on the subject. An experiment illustrating Faraday's theory was also made by Page[5] in which the preliminary heating of the rocker is dispensed with and replaced by the local heating at the points of contact produced by the passage of an electric current during the experiment. Briefly stated, the view now generally held regarding the mode of action of the apparatus is as follows:—

(1) The rocker moves under the action of gravity (its own weight), resting alternately upon two ridges of support.

(2) This movement is sustained by the local periodic expansions of the lead block (which has a high rate of expansibility with increasing temperature and a low heat-conductivity), in consequence of the intermittant contact with the heated metal. The forces brought into play by these expansions do work and supply the energy requisite for the maintenance of the motion. Davis[6] examined the subject mathematically in a paper which we shall have occasion to refer to again, and came to the conclusion that the foregoing explanation of the maintenance of the motion was adequate.

The present work was undertaken with a view to determine by direct experimental study (and not merely on *a priori* suppositions) the manner in which the heated bar vibrates and gives rise to musical tones when placed in contact with the cold metal. The

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[1] Faraday: Experimental Researches in Chemistry and Physics, page 311.

[2] Forbes: Phil. Mag., Vol. IV, pp. 15, 182, 1834.

[3] Seebeck: Pogg. Ann., Vol LI, p. 1, 1840.

[4] Tyndall: Sound, third edition, page 52, ed. 1875; Phil. Trans. Roy. Soc., Parts I–II, 1854.

[5] Page: Silliman's Journal, p. 105, 1850.

[6] Davis: Phil. Mag., Vol. XLV, p. 296, 1873.

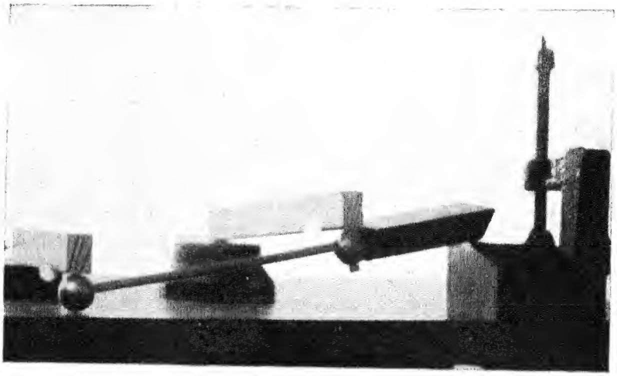


Fig. 1

Trevelyan Rocker arranged to photograph Vibration Curves.

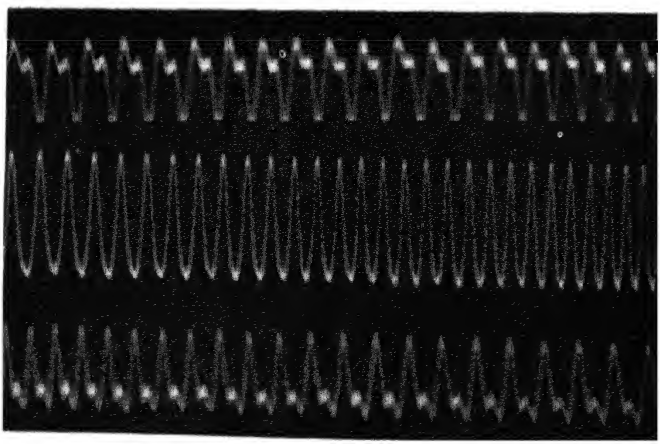


Fig. 2

Fig. 3

Fig 4

Vibration Curves showing effect of Asymmetric Pressure.

*

outcome of the research has been to show that the view usually held regarding the mode of action of the apparatus requires to be modified in essential particulars.

II.—The Vibration-Curves of the Trevelyan Rocker.

The motion of the rocker in the later stages of the experiment, when it gives rise to musical tones, is of quite a small amplitude and requires to be highly magnified to be satisfactorily observed or photographed. The method adopted by the writer has proved itself to be exceedingly simple and effective in practice and is shown in fig. 1 (Plate VIII). A sewing needle is placed horizontally on the curved upper surface of a small metal block which is firmly attached to the rocker. When the rocker vibrates, the needle rolls between this surface and a small bar of wood which is placed lightly resting on the upper surface of the needle, the other end of the bar being supported at a suitable level above the surface of the table. The rotation of the needle is indicated by a mirror attached to it, on which is incident a pencil of light from a pin-hole illuminated by the electric arc. The light reflected from the mirror is focussed by a lens upon a moving photographic plate or by a suitable arrangement can be projected upon a screen to make the form of the vibration-curves visible. Magnification of the movement of the rocker ranging from 1,000 to 20,000 times may be readily obtained in this way.

By fixing the metal block on which the needle rolls firmly to the rocker itself, the motion of the later is readily observed. On the other hand, the block may be fixed, if desired, at any position on the handle of the rocker so as to enable the motion of the latter to be determined. The motion of the upper surface of the ball terminating the handle of the rocker may be similarly observed by a rolling needle and mirror arrangement. Simultaneous observation is also possible of the motion of the end of the handle and of the rocker itself, or of any intermediate point on the former.

III.—Description of the Observed Phenomena.

Using the method described above, numerous observations were made and photographs secured of the mode of vibration of the heated rocker under various conditions, and several quantitative determinations of the frequency and amplitude of vibration

were also obtained. Two different rockers of the usual prismatic form, one rather massive and another of medium size, were used, and some observations were also made with a light rocker having a flat lower surface (instead of a grooved edge). Different lengths and diameters of the handle-bar were employed and the effect of loading the handle-bar at different points was also investigated. Heating the rocker to a greater or less extent, and especially the effect of exerting pressure on the surface of the rocker or on the handle were studied. Observations were also made using a block of rock-salt (which is recommended by Tyndall) instead of lead as the support for the heated rocker. It would be unprofitable to attempt to describe everything that was noticed in the course of the experiments which extended for over four months. *The main result which emerges* from the work is this, that the motion of the rocker, especially in the cases in which it gives rise to musical tones, is *very far indeed from being a simple rocking movement; the elastic vibrations of the rocker and handle-bar play an important part*, and so far as can be ascertained, *entirely replace any such simple rocking movement as may occur in the earlier stages of the experiment.*

The photographs reproduced as figs. 2 to 19 in plate will give an idea of the complexity of the vibration curves. These were all secured with the massive rocker. The noteworthy feature is that the vibrations, when steady, are generally accompanied by harmonic overtones which are of greater or less intensity according to circumstances. In the initial stages, when the temperature difference is very great, a coarse rocking movement of very large amplitude may be observed, but this is of low frequency and hardly gives rise to any audible musical tone. At a slightly later stage, the motion becomes of smaller amplitude, accompanied by partials evidently due to the elastic vibrations of the system and is often rather irregular in character. At a still later stage, the vibrations become very regular and are of markedly musical character. Figs. 5 to 9 represent the successive stages, the vibrations being those of the *end* of the handle bar which was 23 cm. long and 6·5 mm. in diameter. Figs. 10 and 11 are similar photographs with a handle bar 14 cm. long and figs. 12 and 13 with one of 9 cm. length only, the diameter of the handle bar being the same throughout. It is found that the harmonics tend to disappear in the last stages of the motion. Figs. 14, 15 and 16,

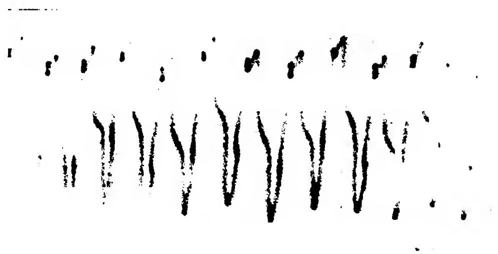


Fig 5

Fig 6

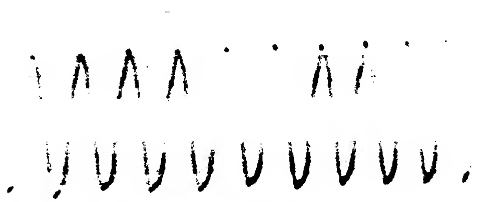


Fig. 7

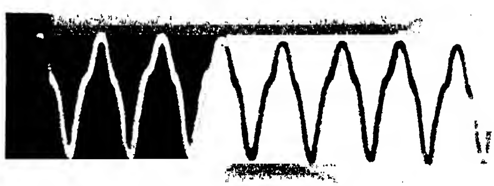


Fig 8

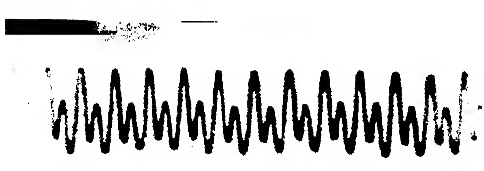


Fig 9

The successive stages of vibration of the end of the handle-bar, 23 cm long and 6 5 m m in diameter.

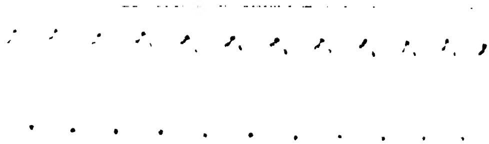


Fig 10

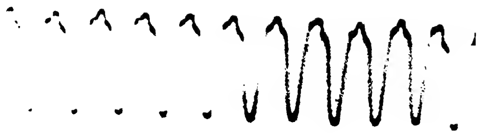


Fig 11

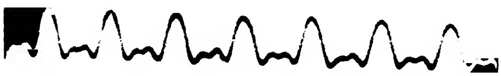


Fig 12

Fig. 13

Fig 14

Fig 15

Fig 16

Fig 17

Fig 18

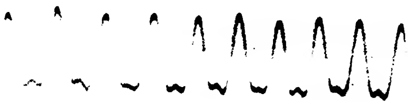


Fig 19.

Vibration Curves of the Trevelyan Rocker.

however, give the vibration curves of the block when firmly clamped to the rocker itself, the handle bar being the longest of the three. It will be seen that the motion of the rocker is, in this case, of much smaller amplitude than that of the end of the handle bar and is rather irregular in character, and is also accompanied by over-tones of high frequency which are not perceptible in the motion at the end. Figs. 2, 3, 4 illustrate an interesting effect. Fig. 3 shows the motion at the end of the handle, when the rocker was vibrating freely. Fig. 2 when the rocker was pressed down by a pointed rod so that the pressure on the lead block was greater on one of the two edges of the groove. Fig. 4 when the pressure was similarly greater on the other edge of the groove. It is seen that the phase of the harmonic is reversed in fig. 4. Figs. 17, 18 and 19 illustrate the motion at the end of the handle bar when the lead block was replaced by a block of rock salt.

The following significant facts were noticed in the course of investigation:--

1. By simultaneous observation of the motion of the rocker and of different points on the handle bar, it is found that the vibration curves at different points differ greatly in amplitude and character. The motion at the end of the handle is always of greater amplitude and indeed sometimes much greater than that of the rocker itself. In the course of vibration, the system divides itself into segments. The points of minimum motion are not however absolutely at rest but show the overtones very prominently. The relation between the movements at different points is not constant but changes when the pitch of tone emitted alters.

2. The musical tone given by the rocker when it is vibrating freely does not change pitch quite gradually but generally passes more or less abruptly from one pitch to another, the amplitude of vibration also changing suddenly at the same time.

3. The effect of pressing the rocker down on the block is always to raise the pitch; but in some cases, the amplitude of oscillation of the rocker actually increases with the rise of pitch, and in other cases decreases. When the amplitude increases with the rise of pitch as the rocker is pressed down, the loudness of the tone markedly increases.

4. Even a slight pressure on the end of the handle bar which rests on the table sometimes results in a marked rise in the pitch

of the tone of the rocker. For instance, in the case of the massive rocker with the longest handle, a pressure of only 30 gms. weight resulted in an increase of pitch from 140 to 180 vibrations per second.

5. When the rocker vibrates freely, the lower and higher limits of pitch of the musical tone which it gives rise to are determined by the mass of the rocker as well as by the length of the handle bar and its cross section. The shorter or the thicker the handle bar, the higher the pitch of the musical sound when it begins and the higher when it ends. Diminishing the mass of the rocker raises the pitch of the tone. The lower end of the range of pitch covered by the musical tones of the freely vibrating rocker is generally about the same as or a little less than the pitch of the tone given by its elastic vibrations in the gravest mode. The higher end of the range of pitch is generally rather ill-defined.

IV.—A CRUCIAL TEST OF DAVIS'S THEORY.

The fundamental assumption on which Davis (in the paper already cited) bases his investigation is that the motion of the heated rocker is a simple rocking movement about the edges under the action of its own weight. The observations and photographs of the vibration-curves of the rocker discussed above render this view untenable and tend to show indeed that it is the elastic vibrations of the system that are maintained and that determine the pitch of the tone emitted. The measurements of the amplitude and frequency of the rocker furnish a crucial test of Davis's theory which we now proceed to discuss.

The motion of the rocker alternately about its two edges under its own weight is not an isochronous oscillation. Its amplitude decreases (in the absence of a maintaining force) continually on account of the loss of energy resulting from the impacts on the edges; and the intervals between successive impacts also continually decrease. The relation between the period and the amplitude of the movement of the rocker at any stage may be determined from the equation of motion which to a first approximation is

$$MK^2 \frac{d^2\theta}{dt} = -Mga$$

where M=mass of the rocker,

K=radius of gyration about either of the edges of support, and $2a$=Distance between the edges.

Hence, integrating,

$$K^2 \int_{t=0}^{t=T/4} \frac{d^2\theta}{dt^2}\,dt = -ag \int_{t=0}^{t=T/4} dt$$

remembering that, when

$$t=0, \frac{d\theta}{dt}=\omega$$

$$K^2\omega = ag\ {}^{T}/_{4} \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now, θ being the greatest angle which the plane containing the edges makes with the horizontal surface of the block

$$\theta=\frac{R}{2a}, \quad R \text{ being the amplitude.}$$

Hence, since ω increases from 0 value to ω in the interval ${}^{T}/_{4}$

$$\frac{\omega}{2}=\frac{2R}{aT} \quad \text{or} \quad \omega=\frac{4R}{aT}.$$

Substituting for ω in (1)

$$T^2=\frac{16RK^2}{a^2g} \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

On the left hand side of this equation, the period of oscillation T of the rocker may be determined experimentally by finding the pitch of the tone given by it with the sonometer or by recording its vibration curves simultaneously with those of a tuning fork. On the righthand side, all the quantities are known except the value of R which may be determined from the observation of the magnified amplitude of the vibration of the rocker. For if I be the movement of the spot of light which we observe and d be the distance of the focus from the lens, r the radius of the rolling needle, then the displacement of the upper surface of the curved mass of metal upon which the needle rolls is given by $\frac{rI}{2d}$.

Hence, knowing the distance of this surface from the axis of rotation of the rocker and the distance between the two ridges, we may deduce the magnitude of the vertical displacement of the free edge from the surface of the block. Thus it is possible to determine directly the quantities on either side of equation (2) and to test its

validity and therefore also the validity of Davis's theory. In the table below, the first column gives the serial number of observation (the handle of the rocker being of different length in some of these cases). The second column gives the frequency of the tones as determined by the sonometer. The third gives the value of the magnified motion (1), the fourth column gives the displacement of the upper surface of the curved mass of metal, the fifth column gives the vertical movement R of the edge of the rocker, and in the sixth column we have the value of the frequency calculated from the equation (2). It will be noticed that the observed and calculated frequencies differ very widely, and it is clear, therefore, that the frequency of the oscillation is not determined by the motion of the rocker under the action of its weight. To exhibit this even more clearly, column 7 of the table shows the value for I, the movement of the spot of light, that would be necessary to give the observed frequency of oscillation if equation (2) were to hold good. It will be found on comparison of column (3) and (7) that the observed oscillation is of far smaller magnitude than deduced from the observed frequency.

DIMENSIONS OF THE ROCKER.

$2a$ = Distance between the ridges
= 0·6 cm.
Base of the triangular section
= 5 cm.
Length of each side of the triangle
= 3.4 cm.
Height of the C.G. above a line joining the ridges
$= b =$ 1·41 cm.
$K^2 = 3{\cdot}47$
(Radius of the needle r = 0·042 cm.
d = 98 cm.)

1	2	3	4	5	6	7
No. of Observations.	Frequency by sonometer.	I (in cm.).	Displacement $=\frac{rI}{2d}$ cm.	Vert. displacement of the ridge. (in cm.).	Frequency (calculated).	Value for I that would fit in equation (2).
1	115	0·75	0·00016	0·000018	301	5·05 cm.
2	121	0·70	0·00015	0·000017	311	4·7 cm.
3	130	0·64	0·00013	0·000015	331	4·0 cm.
4	136	0·60	0·00012	0·000013	342	3·62 cm.
5	145	0·50	0·00010	0·000011	381	3·20 cm.
6	162	0·40	0·00008	0·000009	420	2·6 cm.
7	180	0·20	0·00004	0·000005	595	2 cm.

The foregoing conclusions are not invalidated on taking into account the maintenance of the motion by the periodic expansions and contractions of the block of lead on which the rocker rests. For, to a first approximation, the period of the maintained oscillation must be the same as that of the unmaintained motion having the same amplitude. It can readily be shown that according to Davis's theory, the actual expansions and contractions of the lead block necessary to maintain the oscillation are small compared with the motion of the free edges of the rocker. This may be done in the following way:—

Let ω and ω' be the angular velocities just before and just after impact of the rocker upon one point of the lead block (say O). Then the relation connecting them is[1]

$$\omega=\frac{K_1^2+b^2+a^2}{K_1^2+b^2-a^2}\times\omega',$$

K_1 being the radius of gyration of the rocker about the centre of gravity. If the vibrations are continuous, the angular velocity acquired just before the next impact upon the other point of support (say O') must be equal to $-\omega$. Immediately after the impact upon O, the point begins to rise. Suppose, it has risen through a

[1] Davis, loc. cit.

height h in time t. Then it can be shown that the equation of motion of the rocker is

$$\left(K_1^2 + a^2 \times b^2\right)\frac{d^2\theta}{dt^2} = -a\left(g + \frac{d^2h}{dt^2}\right)$$

that is
$$K^2\frac{d^2\theta}{dt^2} = -a\left(g + \frac{d^2h}{dt^2}\right)$$

where K_1=radius of gyration about an axis passing through the centre of gravity,
and K=radius of gyration about one of the edges of the rocker.

Hence integrating and noticing that

when
$$t=o,\ \frac{d\theta}{dt} = \omega'$$

we have
$$K^2\frac{d\theta}{dt} = -agt - a\frac{dh}{dt} + K^2\omega'$$

Integrating again and noticing that

when
$$t=o,\ h=o,\ \text{and}\ \theta = \frac{H}{2a}$$

(H being the height through which O rises),

we have
$$K^2\theta = -\frac{agt^2}{2} - ah + K^2\omega' t + K^2\frac{H}{2a}.$$

Now, just before the next impact O will have risen to a height H, and O' will have sunk down to its original position. Therefore the value of θ and $\frac{d\theta}{dt}$ will be $-\frac{H}{2a}$ and $-\omega$ respectively. The time for which the rocker is in contact with the block being $T/2$ where T=period of vibration, the above relation takes the form,

$$K^2\frac{H}{a} = \frac{agT^2}{8} + aH - K^2\omega'.T/2$$

whence
$$\omega' = \frac{a^2gT^2 + 8H(a^2 - K^2)}{4aK^2T}$$

or
$$e.\omega. = \frac{a^2gT^2 + 8H(a^2 - h^2)}{4aK^2T}.$$

where,
$$e = \frac{K_1^2 - a^2 + b^2}{K_1^2 + a^2 + b^2}.$$

Substituting in this equation, the value for ω as obtained before, viz.

$$\omega = {}^{agT}/_{4}K^2$$

$$H = \frac{a^2 g T^2 (1-c)}{8(^2 - aK^4)} \dots\dots\dots\dots\dots\dots \quad (3)$$

From equations (2) & (3) we have

$$\frac{R}{H} = \frac{K^2 \cdot a^2}{2K^2(1-c)} = \frac{3{\cdot}47 - {\cdot}09}{3{\cdot}47 \times {\cdot}1} = 9{\cdot}7$$

The free motion of the edges of the rocker required on Davis's theory is thus about ten times the expansion of the supporting block necessary to maintain it, and the influence of the latter on the period of the motion under gravity is therefore quite negligible. Thus we finally arrive at the conclusion that the observed amplitudes and frequencies are not connected together in the manner necessary for the validity of Davis's theory, and the discussion supports the view that the frequency of oscillation of the rocker is determined principally by its elastic vibrations.

The two facts now remaining to be explained are (1) the rise of the pitch of the tone of the rocker as the motion dies away and (2) also, the rise of pitch resulting from exerting pressure on the rocker. It does not seem difficult to reconcile these with the view that the pitch is determined by the elastic vibrations of the system. For, the contact of the rocker with the lead block at one end and of the handle with the table at the other end no doubt operates as a constraint tending to raise the pitch of the elastic vibrations to a greater or less degree depending on the effectiveness of the constraint. Exerting a pressure on the rocker by the point of a rod resting upon it would similarly operate as a constraint raising the pitch of the vibration to quite a considerable extent.

V.—Summary and Conclusion.

The paper describes an experimental study and a theoretical discussion of the vibrations of the Trevelyan Rocker, and the manner in which the musical tones given by the rocker arise. A method is described by which the vibration curves at different parts of the rocker may be observed and photographed and numerous illustrations are reproduced. The theory of the rocker worked out mathematically by Davis is critically discussed, and

it is shown that the basis on which it rests, namely, that the motion of the heated bar is a simple rocking about the two edges under the action of gravity is contradicted by observations on the amplitude and frequency of the vibration of the rocker. The outcome of the research is to show that the musical tones principally arise from the elastic vibrations of the system composed of the rocker and its handle, and its pitch is determined by these vibrations. As the vibrations occur under constraint, the pitch may vary to some extent with the experimental conditions. The effect of pressure on the rocker in raising the pitch is similarly explained.

In conclusion the author wishes to record his best thanks to Prof. C. V. Raman who suggested the investigation and kindly provided all facilities for work at the University College of Science.

Dated Calcutta,
The 18th October, 1920.

XII. Some New Illustrations of Optical Theory by Ripple Motion.

By Rajendra Nath Ghosh, M.Sc.

(Plate XI.)

CONTENTS.

I.—Introduction.

Since Vincent published his well-known photographs of ripples presenting analogies with various optical phenomena, the subject has continued to attract attention from other workers who have used the same or different methods. A fairly comprehensive bibliography of the literature will be found in a recent paper by Watson and Shewhart.[1] In the present paper will be described some further studies in the same field carried out by the author while working at the Indian Association for the Cultivation of Science, Calcutta. The research was undertaken with the aid of a new apparatus which is very compact and serves as a convenient method of producing ripples and observing them by intermittent illumination, the frequency and amplitude of the ripples being both capable of being rapidly altered without interference with the stroboscopic arrangement. The design of the apparatus is due to

[1] Physical Review, February 1916, page 226.

Prof. C. V. Raman and is somewhat similar to that described by him in a recent paper on the experimental study of vibrations.[1]

II.—DESCRIPTION OF THE NEW RIPPLE APPARATUS.

Fig. 1 in Plate XI is a picture of the apparatus used for exciting ripples, part of the ripple tank also appearing in the photograph reproduced. An electric motor causes a horizontal pivoted rod to oscillate rapidly in a vertical plane, a dipper hanging from the end of this rod exciting the ripples on the surface of the liquid contained in the tank seen to the right in the picture. The motor also carries a card-board disc with a sector cut out in it through which the full beam of an electric lantern can pass once in each revolution of the motor. The beam of light is reflected by means of mirrors and comes up through the glass bottom of the tank, the liquid used in it being generally water. The ripples are then seen on a screen held vertically above the tank. They may, if desired, also be projected horizontally on an ordinary lantern screen by using a mirror held at 45° above the surface of the tank to reflect the light at right angles to its path. Since the flashes of light are synchronous with the oscillations of the dipper, the ripples appear perfectly stationary. The frequency and wave-length of the ripples may be altered without interfering with the running of the apparatus by merely increasing or decreasing the speed of the motor by adjustment of a rheostat kept in circuit with it. The amplitude of the ripples excited may be adjusted to any desired value and kept constant by moving the crank-pin on the shaft of the motor to the position necessary and then fixing it. As the arrangement is very robust, any desired form of dipper may be put on without prejudice to the working of the apparatus. Since the full beam of the lantern can pass through the aperture in the stroboscopic disc, there is an abundance of light for all purposes. The device for altering the wave-length of the ripples is very useful in illustrating the theory of interference and diffraction phenomena on the screen by lantern projection. The ripples are seen perfectly at rest and can be photographed from the screen with any ordinary camera.

[1] Physical Review, Nov. 1919. See also these Proceedings, Vol. VI, Plate V.

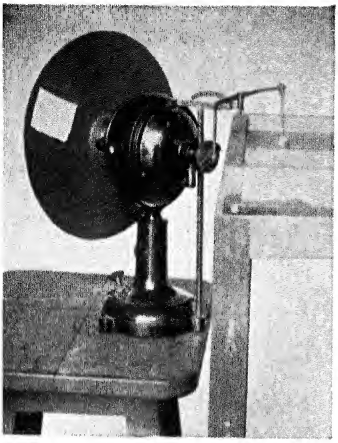
g. 1



Fig. 3

g. 2



Fig. 4

Ripple Apparatus and Photographs.

III.—Interferences near Caustics and the Theory of the Rainbow.

The geometrical theory of the rainbow given by Descartes accounted for the principal bow qualitatively. The more complete theory (involving interferences) was given by Young who thus explained the supernumerary bows. Airy showed that the wave-front emergent from the drop has the form of an inflected curve, and according to his calculations, the direction of the principal maximum of intensity is slightly different from the normal to the inflectional tangent, which is the direction (accordingly to the geometrical theory) in which the least deviated ray travels. The supernumerary bows are the interference fringes formed alongside the caustic, and they decrease gradually in width and intensity as we move away from the principal fringe.

The theory of the rainbow may be beautifully illustrated by means of ripples. For this purpose, the dipper used to excite the ripples is given the form of the inflected wave-front emergent from the drop, of course on a much larger scale. The dipper should be carefully shaped by drawing the desired form of curve and bending a strip of brass or copper to its shape, and it should just touch the surface of the liquid. The ripple as it starts out from the dipper has the form of an inflected curve, but in proceeding outwards it doubles up and forms two superposed ripple-trains which interfere, giving maxima and minima of gradually decreasing width and intensity. This effect is clearly shown in fig. 2 in Plate XI. The photograph clearly shows how the ripples from the concave half of the strip converge to a train of foci, then diverge again, and ultimately become superimposed on the ripples diverging from the convex half, giving interferences. The great width and clearness of the principal fringe, and the weaker and narrower fringes lying on one side of it are clearly shown. In agreement with theory, the minima are practically lines of zero amplitude, and the direction of the maximum disturbance is also deviated a little away from the normal to the curve at the point of inflection and towards the first minimum lying on one side of it. An interesting feature will also be noticed in the photographs that the ripples in the two regions of maximum lying on either side of each minimum are displaced relatively to each other. This occurs to a

greater or less extent in all cases of interference, and is closely connected with the question of energy flow in the field.

The intensity at any point in the interference field due to an inflected wave-front can be readily calculated by elementary methods without carrying out the usual elaborate integration over the wave-front first worked out by Airy.

Let Oy be the normal to the inflectional tangent xOx', the equation of the wave-front being $y=Ax^3$. Let P and P' be corresponding points at which the wave-normals are parallel to each other and inclined at an angle θ to the y-axis. We may assume without appreciable error that the effects of different parts of the

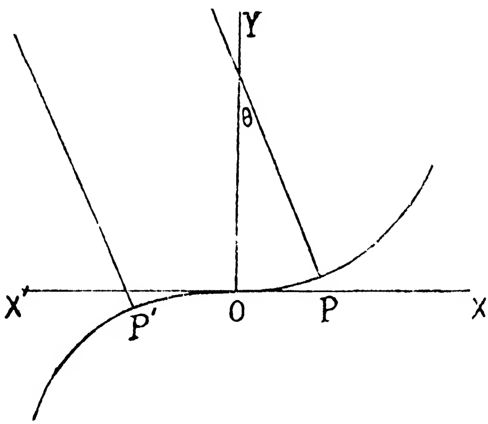


FIG. 5.

wave-front are propagated as rays in the direction of the different wave-normals in accordance with the laws of geometrical optics. The curvatures of the wave-front at P and P' are equal and of opposite sign, and hence at a sufficiently great distance the effects contributed in the direction of the wave-normals at these points will be of equal amplitude. These effects are proportional to a_1 where

$$a_1^2 = 1\Big/\frac{d^2y}{dx^2} = 1\Big/6Ax = \pm\left(\frac{3A}{\theta}\right)^{\frac{1}{2}}\Big/6A \;\ldots\ldots\ldots\ldots \quad (1)$$

On taking the square root, the value a_1 corresponding to the negative sign becomes imaginary. We may interpret this as a gain of phase of $\pi/2$ which occurs in the rays from the part of the wave-

front which changes the sign of its curvature in passing through a focus. The numerical value of a_1 is

$$(12A\theta)^{-1/4}.$$

The path difference of the rays proceeding from P and P' can then be easily shown to be

$$\frac{2\pi}{\lambda}\left[\frac{4}{3}\theta\left(\frac{\theta}{3A}\right)^{\frac{1}{2}}\right]-\frac{\pi}{2}.$$

The amplitude of the resultant of the two will be

$$a_2 = 2(12A\theta)^{-1/4}\cos\left\{\frac{\pi}{\lambda}\cdot\frac{4}{3}\theta\left(\frac{\theta}{3A}\right)^{\frac{1}{2}}-\frac{\pi}{4}\right\} \quad \ldots\ldots\ldots\ldots \quad (2)$$

On squaring this, we get the intensity in the direction θ, and it can easily be shown that equation (2) gives results which, for positive

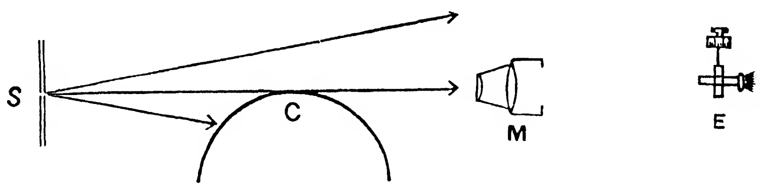


FIG. 6.

values of θ greater than a certain very small limit, are for all practical purposes the same as those given by Airy's integral. In fact the result given by (2) is identical with that obtained from Airy's integral by the method of semi-convergent expansions.

The most convenient experimental arrangement for observing on the optical bench the different steps in the transformation of an inflected wave-front to a cusped wave showing the interferences alongside the caustic is that indicated diagrammatically in fig. 6.

S is the slit which is the source from which light diverges and falls upon the reflecting cylinder C. M is a short focus lens which is placed so that its focal plane is well in advance of the edge of the cylinder C grazed by the incident rays. The rays after reflection at the surface of the cylinder form a virtual caustic within the cylinder. After passing through the lens they form an inflected wave-front which in the region behind, forms a cusped wave-front

and shows the interferences alongside the caustic very well.[1] By interposing a wire of greater or less diameter immediately behind the lens, different portions of the inflected wave-front can be cut off. As the wire is moved across the field from one side to the other, its shadow moves in the interference-pattern towards the caustic and then *retraces its path*, the fringes practically disappearing from the region covered by the shadow,[2] showing thereby clearly that the inflected wave doubles up and forms a cusped wave-front in the course of its propagation and thus gives rise to interferences.

The formula (2) given above for the disturbance in the interference-field due to an inflected wave-front was tested experimentally *with the ripple apparatus* by measuring the angular positions of the minima of disturbance relatively to the inflectional tangent, and also calculating the same from the observed wave-length of the ripples and the known form of the dipper. The results are shown in the table given below :—

TABLE I. $\lambda = 1{\cdot}07$ cm. $A = 0{\cdot}45$.

Minimum No.	1	2	3	4	5
Pontoon observed ..	66°	50°	33°	29°	21°
,, calculated ..	68°	52°	40°	28°	23°

The agreement is fairly satisfactory.

IV.—ANAMOLOUS PROPAGATION NEAR FOCI.

Gouy discovered in 1890[3] that a spherical wave gains in phase by half a wave-length in passing through a focus. His experiments were repeated by Fabry[4] and Zeeman. Later Sagnac[5] and Reiche again performed the experiment. Sagnac on the basis of diffrac-

[1] See N. Basu, Phil. Mag., Jan. 1918, page 92.

[2] The fringes elsewhere are also disturbed by the diffraction effects due to the wire, but not to such an extent as to obscure the general nature of the case. Some interesting effects are observed when the wire just coincides with the point of inflection in the wave-front.

[3] Gouy, Ann de Chim et de Phys., 6th series 24, pp. 145–213.

[4] Fabry, Journal de Phys., 3rd series 2, p. 22 (1892).

[5] Sagnac, Journal de Physique, Oct. 1903.

tion theory explained the gain of phase as due to the oscillation of phase and intensity; Reiche[1] on the electromagnetic theory showed that the electric vector in passing through the focus changes sign while the magnetic vector remains unchanged, indicating a gain of phase of half a wave length in the passage through the focus. Since it is now established that a wave gains in phase in passing through a focus, it appeared worth while to see whether it is possible to detect the phenomenon in the case of ripples as well. With this object, the dipper was made into the shape of an arc of a circle of radius 3·5 cm. When a plane wave passes through a lens, it is transformed into a concave wave which first converges to a focus, and then diverges into the surrounding medium in the form of convex waves. The waves generated by the dipper behave similarly. Fig. 3 in the plate shows the effect. Careful observation shows that the wave lengths near the focus are slightly longer than the others. This can be interpreted as equivalent to an acceleration of velocity near the focus, so that the waves gain a certain fraction of wave-length in phase in passing through it. Using very small amplitudes of excitation of the ripples so as to avoid errors due to the alteration of the velocity of the ripples which occur at larger amplitudes, a large number of measurements were made with the ripple apparatus, the total space occupied by three waves at the focus, and three waves on either side of it being determined. The former was found to be greater than the mean of the two latter by about quarter of a wave-length, which is what we should expect in the case of cylindrical waves in the optical case. It may be remarked that this gain in phase in passing through a focus is also indicated by the good agreement between theory and experiment in the case of the rainbow (dealt with in the preceding section) where the phase-change was taken into account in finding the positions of the minima of illumination.

It may be remarked in passing that fig. 3 also clearly shows the diffraction-effects in the neighbourhood of the focus. It is precisely because of diffraction that it is necessary to take into account a sufficient number of waves on either side of the focus in order to determine the gain in phase.

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[1] Reiche, Ann. d Physik, April and June 1909. (An account is given in Wood's Optics, page 263.)

V.—LAMINAR DIFFRACTION.

In some recent studies of laminar diffraction phenomena,[1] it has been found that the edges of thin laminae often diffract light in a strongly unsymmetrical manner, a great deal more light being scattered towards the retarded side of the wave-front than on the other. It was thought that it would be of interest to try and illustrate this phenomenon with the aid of the ripple apparatus. A dipper was used of the form of a long  so as to imitate the discontinuous form of wave which presumably arises when a plane wave passes through a semi-infinite transparent lamina with a sharp straight edge. The result obtained is illustrated in fig. 4 of the Plate. It will be seen that the disturbance arising from the region of bend in the dipper is overwhelmingly large to the region on the right which corresponds to the retarded side of the wave-front, while on the left of the normal drawn to the line of the dipper, the ripples are almost perfectly straight with hardly any sensible disturbance. Reversing the position of the dipper so that the retarded side of the ripple was on the left immediately resulted in a corresponding reversal of the pattern showing that the effect was genuine.

VI.—SUMMARY AND CONCLUSION.[2]

The paper describes the construction and use of a very simple and compact apparatus which enables ripples to be excited on the surface of a liquid and observed stroboscopically, the frequency of the ripples being capable of rapid alteration while the apparatus

[1] P. N. Ghosh, Proc. Roy. Soc. London, A series, Vol. 96, 1919, page 261; Proc. Indian Assoc. for the Cultivation of Science, Vol. VI, part I, page 52.

[2] It may be of interest to note here that since the return of Mr. R. N. Ghosh to Allahabad to join his appointment at the Ewing Christian College, the work with the ripple apparatus used by him has been continued at this laboratory. Mr. Goverdhan Lal Datta has shown that the apparatus can be used to exhibit the analogy of the effect of groove form on grating spectra, and has succeeded in constructing a ripple grating which concentrates nearly all the energy in a single spectrum. Mr. Datta has also studied the character of interference and diffraction phenomena in the *immediate neighbourhood* of the sources which had not previously received adequate attention. Some very interesting results have also been obtained by Mr. J. C. Kameswara Rao on the forms of ripples of large amplitude. These are described in a paper appearing later in this volume. The work done has demonstrated the convenience and utility of the apparatus in the exact study of ripple motion. (C. V. R., 26 Oct., 1920.)

is running without interfering with the stroboscopic arrangement. The amplitude of the ripples excited can also be varied and kept strictly constant.

With the help of the apparatus the following new illustrations of optical theory by ripple motion have been worked out:—(1) Interferences near Caustics and the theory of the Rainbow; (2) Anamolous Propagation near Foci; (3) Unsymmetrical Types of Laminar Diffraction, and photographs of these cases are reproduced with the paper.

In conclusion, the author wishes to record his grateful acknowledgments to Prof. C. V. Raman for his suggestions and constant encouragement during the progress of this research.

DATED CALCUTTA,
The 15*th of June*, 1920.

XIII. The Theory of Impact on Elastic Plates.

By K. Seshagiri Rao, B.A. (Hons.), Research Scholar in the Indian Association for the Cultivation of Science, Calcutta

SYNOPSIS.

1. *Hertz's Theory*:—In a paper contributed to the *Physical Review* for April 1920, C. V. Raman has shown how Hertz's theory of the collision of elastic solids may be extended to the case of transverse impact on elastic plates so as to determine the proportion of the kinetic energy of the impinging body transformed to energy of wave motion in the plate. The assumptions forming the basis of this extension of Hertz's theory have been tested experimentally by the present author and it is shown how the residual discrepancies between observation and the results given by Raman's theory in the case of relatively thin plates may be explained.

2. *Duration of Impact*:—Except in the case of relatively thin plates the duration of impact is found to be substantially the same as that given by Hertz's theory of impact on an infinite mass of solid with a plane face; however, when the thickness of the plate is diminished to nearly the critical value at which the whole of the energy of the impinging body is transformed into energy of wave-motion, experiments show an appreciable increase in the duration of contact. It is shown that this is what might reasonably be expected on theoretical grounds.

3. *Apparent Coefficient of Restitution*:—Allowing for a small proportion of energy dissipated otherwise than in the production of wave motion in plate, and also making a correction for the increase of duration of impact referred to above, good agreement is found between theory and experiment for the values for the apparent coefficient of restitution. The increase of the coefficient of resti-

tution for decreasing velocity of impact is quantitatively explained.

4. *Acoustic Applications*:—The theory with suitable modifications appears to be capable of being applied to the problem of the analysis of the vibrations of bells or plates excited by impact.

I.—INTRODUCTION.

The mathematical theory of the collision of elastic solids given by Hertz is primarily applicable only to cases in which the shape of the impinging bodies and their velocity of collision is such that a negligible proportion of their kinetic energy is transformed into energy of wave motion in the solid or dissipated in producing quasi-permanent deformations. It is naturally of interest to determine how Hertz's theory has to be modified in various cases where such transformations of energy do occur to an appreciable extent. In a paper recently contributed to the *Physical Review* for April 1920, Prof. C. V. Raman has shown how the problem of transverse impact of a solid with curved faces on a plane elastic plate may be dealt with by an extension of the method suggested by Hertz, and has given a formula for the apparent coefficient of restitution of the impinging solid from which the proportion of its kinetic energy transformed into energy of wave motion in the plate may be calculated. In this application, it was assumed that the duration of impact was to a first approximation the same as that given by Hertz's theory for impact on the plane face of an infinite mass of solid. From the theory the result was deduced that as the thickness of the plate is gradually diminished, the apparent coefficient of restitution should continually diminish till when the thickness is below a certain critical limit depending upon the elastic constants and upon the mass, dimensions and velocity of the impinging body, the coefficient of restitution should vanish altogether: in other words that the body should then behave as if it were perfectly inelastic and remain in contact with the plate. It is obvious that when this critical limit is reached the duration of impact should become infinite: in other words that the very assumption on which theory itself is based should break down at or near this point. As a matter of fact, observation shows that while the calculated coefficient of restitution is in excellent agreement with the theory for plates of moderate thickness, deviations appear when the thick-

ness is rather near the critical limit and in such cases the observed coefficient of restitution falls less rapidly to zero with decreasing thickness of the plate than the calculated coefficient. It would appear from these considerations that in the case of the relatively thin plates the duration of impact should be somewhat larger than that given by Hertz's theory, and that this should be taken into account in the calculation of the apparent coefficient of restitution and that a better agreement might then be expected between theory and experiment in the case of such plates. The present paper describes work undertaken by the author mainly to clear up the outstanding discrepancies between theory and observation on the lines explained above.

II.—Some Experimental Results.

The materials selected for the investigation consisted of a set of five mild steel plates of thicknesses 3·82, 2·30, 1·81, 1 21 and 0·60 cms. respectively. The impinging body was a sphere of hard steel of diameter 2·70 cms.

The first set of experiments consisted in a determination of the duration of impact between sphere and plate for different velocities of impact. This was done in the usual way by allowing an electric circuit to be completed by the contact between the plate and the ball, the total quantity of electricity which passes from the time they meet till they again separate being measured on a ballistic galvanometer. Without taking into consideration the corrections for self-induction we get the formula

$$Q = \frac{V}{R} T \quad \text{or} \quad T = \frac{RQ}{V}$$

where R is the resistance, Q quantity of electricity and T the time of impact. The correction for self-induction are two in number.

(*a*) The current instead of rising instantaneously to its full value when the ball and the plate meet grows exponentially.

(*b*) When the ball separates from the plate a spark may be produced so that the current does not fall instantaneously to zero.

These corrections can be determined as follows: If a fairly large resistance be included in the circuit we can consider the contact resistance as negligible compared to the first. It follows therefore when the ball impinges the resistance rises to its final

value almost instantaneously. We may therefore treat the circuit as one having constant resistance. The equation of growth will then be the usual expression

$$L\frac{di}{dt} + Ri + V = o$$

or
$$i = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Integrating
$$Q = \frac{V}{R}\left\{T - \frac{L}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right\}$$

$$= \frac{V}{R}\left(T - \frac{L}{R}\right) \text{ if } \frac{R}{L} \text{ is large.}$$

$$\therefore T = \frac{RQ}{V} + \frac{L}{R}.$$

The correction is thus a constant addition to the time. The method of investigating these corrections was the same as used by Sears[1] for the problem of longitudinal impact of rods. The complete connections are shown in the figure.

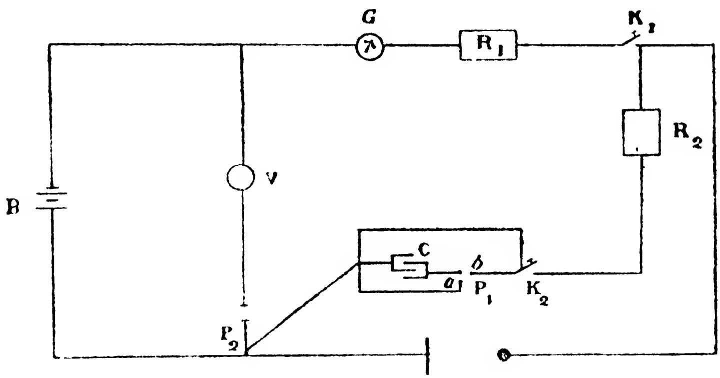

FIG. 1.

B was a battery of 4 volts and *R* was a resistance box from which 5000 ohms were unplugged. *C* was the condenser used in calibrating the galvanometer. With the high resistance used it was found that there was no sparking effect at all while the correction for self-induction was very small.

[1] Proceedings of Cambridge Philosophical Society, Vol. XIV, page 273.

The ball was suspended by a fine covered copper wire which was soldered to it, and which also served to convey the current during the impact. The distance of impact was regulated by holding the ball in position by an electro-magnet, care being taken that there was no leakage through the electro-magnet or through the suspension wire. Both the sphere and the surfaces of the plates on which the sphere impinged were highly polished and cleaned. It was found that concordant results could be obtained only when these were scrupulously clean, the slightest trace of dirt producing great variations in the readings.

For every plate the duration of impact was determined with different velocities. In every case it is found that the duration varies as the fifth root of velocity in the manner required by Hertz's theory. It is also found that the duration of contact is practically the same for the plates having thickness from 3·80 to 1·21 (though a very slight increase is to be noticed for decreasing thickness); but suddenly increases by 10% for the plate having a thickness of 0·6 cms. The table showing the results is given below.

TABLE I.

The height of suspension of the ball 74·35 cms.
Voltage applied 4·23.

Plate No. 1, Thickness 3·82 cm.

Withdrawal a	Galv. Throw θ	$\sqrt[5]{a} \,.\, \theta$
7·0 cm.	97·0 divisions.	157·8
4·2 ,,	108·0 ,,	157·2
2·8 ,,	129·0 ,,	158·5
2·2 ,,	136·0 ,,	158·1

Plate No. 2, Thickness 2·30 cms.

Withdrawal a	Galv. Throw θ	$\sqrt[5]{a} \cdot \theta$
7·3 cm.	105	156·3
5·9 ,,	110	157·0
4·1 ,,	119	157·5
2·7 ,,	130	158·0
1·5 ,,	146	158·0
		Mean 157·6

Plate No. 3, Thickness 1·81 cm.

Withdrawal a	Galv. Throw θ	$\sqrt[5]{a} \cdot \theta$
7·8 cm.	104	156·7
6·4 ,,	109 5	159·0
5·3 ,,	114	159·0
3 6 ,,	122	157·7
2·5 ,,	132	158·3
		Mean 158·2

Plate No. 4, Thickness 1·21 cms.

Withdrawal a	Galv. Throw θ	$\sqrt[5]{a} \cdot \theta$
7·3 cm.	106 0	158·8
3·9 ,,	119·0	164·8
3·1 ,,	129	161·4
2·2 ,,	135	157·8
3·3 ,,	128	162·4
		Mean 160·6

Plate No. 5, Thickness 0·60 *cm.*

Withdrawal a	Galv. Throw θ	$\sqrt[5]{a}\,.\,\theta$
7·6 cm.	118	177·0
6·3 ,,	121	174·0
5·1 ,,	126	174·0
3·7 ,,	134	175·8
2·7 ,,	142	174·5
1·5 ,,	162	175·4
		Mean 175·1

It will be found that the readings in last column are practically constant for each plate and its change is little except for the thinnest plate.

If we take the mass of the plate to be very great and calculate the duration of contact on the basis of Hertz's theory, we find the calculated value agrees very well with the thicker plates while for the thinnest the discrepancy is about 10%.

TABLE II.

Velocity corresponding to 6 cms. withdrawal 21·78 cms./sec.

Calculated from Hertz's theory.	Observed Values of Duration.				
	Plate No. 1.	No. 2.	No 3	No. 4.	No. 5.
$1{\cdot}265\times10^{-4}$	$1{\cdot}256\times10^{-4}$	$1{\cdot}256\times10^{-4}$	$1{\cdot}259\times10^{-4}$	$1{\cdot}281\times10^{-4}$	$1{\cdot}390\times10^{-4}$

The second set of experiments consisted in the determination of the coefficient of restitution for different velocities of impact. On Raman's theory the expression for e the coefficient of restitution is given by

$$e=\frac{fpa^2-kM}{fpa^2+kM}$$

where k a constant depending on the form of the transverse wave in the plate and

$$a^2 = \pi\tau f \sqrt{E/3\rho(1-\sigma^2)}$$

where E = Young's modulus,
τ = duration impact,
σ = Poison's ratio,
$2f$ = Thickness of plate,
ρ = density,
M = the mass of the impinging sphere.

It will be evident from the expression that e varies with τ, i.e. with the velocity of impact. e was determined experimentally for each plate for different impinging velocities and was calculated from the above expression, k being taken to be 0·56. τ, the duration of impact, is taken in the calculation for the first four plates to have given the value by Hertz's theory. For the thinnest plate, the calculated coefficient of restitution depends on whether τ is to be taken the actual value as found in experiment, or the value as given by Hertz's theory. Both the values are given, the former being marked by asterisks.

TABLE III.

Plate thickness	Vel. 25·4		Vel. 21·8		Vel. 18·2		Vel. 14·5		Vel. 10·9	
	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.
3·82 cm. ..	0·97	0·91	0·97	0·91	0·97	0·91	0·98	0·91	0·98	0·92
2·30 ,,	0·93	0·87	0·94	0·88	0·94	0·88	0·94	0·89	0·94	0·89
1·81 ,,	0·88	0·83	0·88	0·83	0·89	0·84	0·89	0·84	0·90	0·84
1·20 ,, ..	0·77	0·70	0·77	0·71	0·78	0·72	0·79	0·72	0·79	0·73
0·60 ,, .	0·31*	0·30	0·32*	0·31	0·33*	0·32	0·35*	0·34	0·37*	0·36
	0·34		0·36		0·38		0·39		0·42	

In considering the degree of agreement of the calculated and observed coefficients of restitution shown in Table III it must be remembered that Raman's formula takes account of only the trans formation of kinetic energy of translation into the energy of wave

motion in the plate and does not take into consideration the energy dissipated in production of permanent deformation in the plate. Even for the very moderate velocities of impact dealt with in the present investigation, some dissipation of the latter kind is evitable, distinct marks being left on the mild steel plates on the result of impact, and we may reasonably expect the observed coefficient of restitution for the thick as well as for the thin plates to be less than the calculated value by some units in the second place of decimals. This is exactly what appears in table III, in the case of the four moderately thick plates, and also for the thinnest plate if we take the asterisked values of the calculated coefficient for comparison with the experimental values.

DATED CALCUTTA,
The 15th of May, 1920.

XIV. On Ripples of Finite Amplitude.

By J. C. Kamesvara Rav, M.Sc., Palit Research Scholar in the Calcutta University.

(Plate XII.)

CONTENTS.

SECTION I.—INTRODUCTION.*

The problem of wave motion on the surface of liquids has for a long time been the subject of both theoretical and experimental study, its practical interest as well as the facility with which the various phenomena connected with it can be observed contributing to the development of the subject. Scott Russel[1] was one of the earlier observers in this field. Soon after Stokes[2] gave a complete theory of the oscillatory waves of finite amplitude and of permanent type. He found that to a second approximation, the velocity of waves does not depend upon the amplitude of the

* *Note.*—The experimental work described in this paper was carried out in June and July 1920. The mathematical theory was presented at a meeting of the Calcutta Mathematical Society on the 4th of September, 1920. The effects have since been observed also with mercury surfaces.

1 Brit. Ass. Rep., Vol. VI, 1844.

2 Camb. Trans. t. VIII (1847) and papers, Vol. I, p. 197.

waves, and as regards form he found that the hollows were no longer similar to the crests, but the height of the latter exceeds the depth of the former and that the crests are narrower than the hollows. For a third approximation he found that the velocity of propagation increases with the amplitude. In his investigations, he did not however take into account the effect of capillarity, but Kelvin[1] solved the problem completely for small amplitudes and proved the existence of a minimum velocity of propagation. Later on Rayleigh[2] studied ripples experimentally and observed them under intermittant illumination by the method of Foucault's test. By this method he measured the surface tensions of some liquids and the effect of contamination on the surface tension of water. Vincent[3] studied ripples experimentally and took many beautiful photographs, illustrating interference, diffraction, etc., and more recently Watson[4] used an improved method of Rayleigh's for the determination of surface tension and viscosity of liquids. Dr. Wilton[5] gave a theory of the form of ripples of finite amplitude and short wave lengths and he showed mathematically that the length of the troughs should be very small compared with the length of the crests, and in fact the form of the troughs according to him have the shape of a cusp (ceratoid), where the curvature is very great. In the case of fairly large wave lengths (say 2·5 cm.) he obtained two forms theoretically, one similar to those of shorter ripples and the other with two crests in each wave length, and suggested (wrongly as it would appear) that the latter form was unstable.

It was with a view to test the accuracy of Wilton's work that the present investigation was first taken up, but in the course of the research, some quite new and interesting results were obtained. It was found that the form of ripples excited by a simple periodic pressure at a point were not simple undulations but with the increase of amplitude and wave length they assumed very complex forms varying with the conditions. The paper describes the effects observed and the mathematical theory is also worked out.

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1 Baltimore Lectures, p. 598; Phil. Mag (4) t. XLVII, p. 374.

2 Phil. Mag. (5) t. XXX, p. 386; Papers t. iii, p. 394.

3 Phil. Mag. Vol. 43, p. 417; 45, 191, 46, 290.

4 Phy. Rev. Vol 12, p. 257 (1901); Phy. Rev. Vol. 15, p. 20 (1902); Phy. Rev. 2nd series, Vol. VII, p. 226 (1916).

5 Phil. Mag. May 1915, p. 688.

SECTION II.—DESCRIPTION OF THE APPARATUS.

The phenomena described in the present paper are best seen by the method of stroboscopic observation. It is possible though somewhat inconveniently to observe them when the ripples are excited in the usual manner by a low frequency fork, which is electrically maintained. A dipper is attached to one of the prongs and touches the surface of the water and when the fork is set in vibration it excites ripples on the surface. The tank containing water is a rectangular vessel of 5 cm. depth, with a glass bottom. To obtain intermittent illumination two slits are attached to the prongs of the fork, so that they allow light to pass through them once for every vibration and this intermittent light is passed through the bottom of the tank. This arrangement, however, gives an insufficient amount of light. In addition to this difficulty it is not possible to control and vary the amplitude of motion of the dipper and its frequency in the manner desirable in the present investigation. These difficulties are eliminated by the use of Professor Fleming's motor vibrator as improved by Professor C V. Raman. The utility of this instrument for the study of vibrations cannot be overestimated.

The apparatus can be seen in fig. 1 of Plate XI. To the axle of the motor a circular wheel is fixed and to this wheel a brass disc carrying a slot and a movable pin is fixed. This pin actuates an oscillating lever, which moves vertically up and down. To the end of this lever a dipper is attached, which excites the ripples on the surface of the liquid contained in the tank. The tank also is shown in the figure. It is a rectangular vessel with a glass bottom. To the other end of the axle of the motor a circular disc with a slot cut in it is attached. Light from an electric arc is allowed to pass through this hole and is reflected up by a mirror through the bottom of the tank and then by a second mirror projected on a screen. For every revolution of the motor, the light passes once, which makes the ripple pattern stationary. The dipper oscillating in the tank makes approximately a simple harmonic motion and to avoid any sideway motion it is curved into a circular arc of correct radius. In order to be sure that the observed phenomena are not due to any harmonics present in the vibration of the dipper, the experiment is also repeated with a dipper attached to the prong of an electrically maintained fork, the intermittent illumation being

obtained by allowing the beam of light to pass through the stroboscopic disk of the motor, the speed of the latter being suitably regulated.[1]

It is also possible to observe the profile of the waves in the following manner. The intermittent flashes of light from the arc instead of being sent up through the bottom of the ripple tank are allowed to fall upon a white cardboard screen, the straight edge of which is viewed by the oblique reflection at the surface of the water contained in the tank. The edge appears in the form of a fixed wave which corresponds more or less closely with the actual wave-profile. The proper positions being chosen for the observer's eye and for the edge of the screen, satisfactory observations of the wave-profile may be made visually. The crests in the wave-profile correspond to the bright lines on the screen and the hollows to dark spaces.

SECTION III.—OBSERVATIONS OF THE FORM OF RIPPLES OF LARGE AMPLITUDE.

When the amplitude is small, we see on the screen for all wave-lengths a series of concentric circles at equal distances, corresponding to single crests in the waves, but when the amplitude is made great (say 2 mm.) the simple character of the waves is at once altered. For values of λ up to about 1·5 cm. the waves remain simple circles, but if the amplitude is made very great the first few waves divide. When λ lies between 1·5 and 3 cm. the circles become double, corresponding to two crests in each wave-length. For still greater wave-lengths the waves become triple and complex. At this stage it may be pointed out, that the word ripples is used in the popular sense and not in the sense used by Kelvin. Some photographs of the wave patterns are given in figs. 1 to 6, Plate XII. The figs. 1, 2, 3 correspond to a constant amplitude, but of increasing wave-length and figs. 4, 5, 6 correspond to a different amplitude and increasing wave-length. It will be seen from the figures that with the increase of wave-length the doubling of the crests becomes more prominent.

[1] The amplitude of the oscillator can be adjusted by altering the position of the pin in the brass slot and the wave length can be altered by altering the frequency of the motor, which can be managed by including a rheostat in the circuit of the motor.

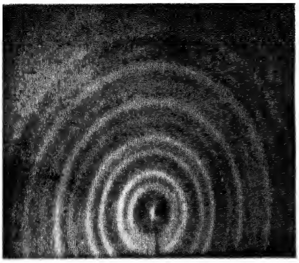


Fig. 1.

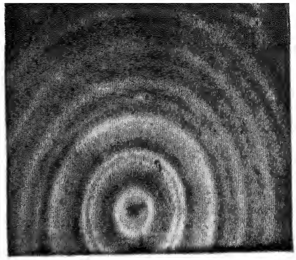


Fig 4.

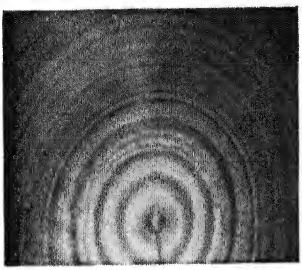


Fig. 2.

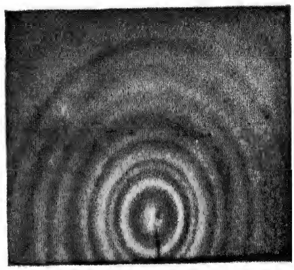


Fig. 5.

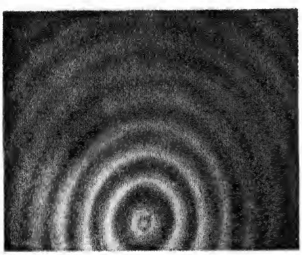


Fig. 3.

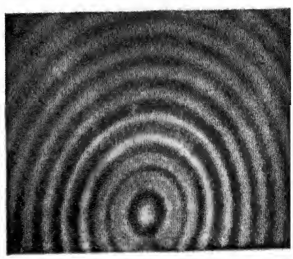


Fig. 6.

Ripples of Finite Amplitude.

Another feature that can be observed in these photographs is the change of form with increasing distance from the source. With increasing distance, the amplitude becomes smaller and consequently the circles become dim and in the case of the double pattern, one of the circles gradually becomes less prominent. This can be seen very well in figs. 3 and 6. Also, at a great distance the curvature becomes smaller, and consequently the bright, narrow circles near the centre broaden out slightly (figs. 1 and 4).

The change of the form of ripples with gradually decreasing depth of the liquid was also studied. Till the depth is reduced to a few millimeters no change is observed; the form then gradually becomes simpler. The critical depth at which the change of form occurs is, as might be expected, less for small wave-lengths and greater for the larger wave-lengths.

A curious anamolous effect is observed if the water be run out of the ripple tank while the dipper continues in oscillation; the wave-form then becomes simple, rather suddenly, and if we stop running out the water at that critical moment, we get sometimes simple crests and sometimes double crests. This effect is observed only when the amplitude is not very large, and is apparently connected with changes in the form of the contact surface between the dipper and the liquid. Usually, however, when the running out of the water from the tank is stopped, the form of the ripples corresponding to the particular depth is at once obtained.

We now proceed to explain the phenomena given in the previous section mathematically. First we take up the simpler case of plane waves.

Section IV.—Mathematical Theory for Plane Waves of Finite Amplitude.

Let the axis of x be in the direction of propagation of waves and the axis of y vertically upwards and the undisturbed level of the liquid as the plane $y=0$, and let the depth of the liquid be h. Then for irrotational two dimensional wave motion, the velocity potential satisfies the equation.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{\partial\phi}{\partial y}=0 \text{ when } y=-h \quad \ldots\ldots\ldots\ldots \quad (2)$$

This condition denotes that the velocity at the bottom is zero and the pressure condition at the surface gives

$$-\frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} = -gy + \frac{\partial\phi}{\partial t} - \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} \quad \ldots\ldots\ldots \quad (3)$$

taking the curvature to be small, which is true for pretty long waves. Since the disturbing force is simple harmonic, we can suppose the following expression for ϕ

$$\phi = \sum_{s=1}^{s=n} a_s \cosh s\, m\, (y+h) \sin sm\, (ct-x) \quad \ldots\ldots\ldots \quad (4)$$

which satisfies conditions (1) and (2).

We have also the condition

$$-\frac{\partial\phi}{\partial y} = \frac{\partial y}{\partial t}, \text{ and for } y = o$$

this gives

$$y = \frac{1}{c}\sum_{s=1}^{s=n} a_s \sinh smh \cos sm\, (ct-x) \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

To determine the constants a_1, a_2, a_3.... we substitute the values of ϕ and y in the pressure equation (3) and equate the coefficients of cos $sm\,(ct-x)$ to zero, we get equations for the determination of c, a_2, a_3 .... etc., a_1 itself being arbitrary. For practical purposes it is sufficient to take only three terms as the coefficients a_4, a_5, etc., become negligibly small.

Thus we can take

$$\phi = a_1 \cosh m\,(y+h) \sin m\,(ct-x) + a_2 \cosh 2m\,(y+h) \sin 2m\,(ct-x) + a_3 \cosh 3m\,(y+h) \sin 3m\,(ct-x)$$

and

$$y = \frac{1}{c}\left\{a_1 \sinh mh \cos m\,(ct-x) + a_2 \sinh 2mh \cos 2m\,(ct-x) + a_3 \sinh 3mh \cos 3m\,(ct-x).\right.$$

Substituting these values of ϕ and y in the pressure equation (3) we get

$$\left(\frac{Tm^2}{\rho c} + \frac{g}{c}\right)a_1 \sinh mh \cos m\,(ct-x) + \left(\frac{4Tm^2}{\rho c} + \frac{g}{c}\right)a_2 \sinh 2mh \cos 2m\,(ct-x) + \left(\frac{9Tm^2}{\rho c} + \frac{g}{c}\right) a_3 \sinh 3mh \cos 3m\,(ct-x)$$

$$= mc\,[a_1 \cosh m\,(y+h) \cos m\,(ct-x) + 2a_2 \cosh 2m\,(y+h) \cos 2m\,(ct-x) + 3a_3 \cosh 3m\,(y+h) \cos 3m\,(ct-x)\,]$$

$$-\frac{m^2}{2}[\{a_1 \cosh m(y+h)\cos m(ct-x)+2a_2\cosh 2m(y+h)\cos 2m(ct-x)$$
$$+3a_3\cosh 3m(y+h)\cos 3m(ct-x)\}^2$$
$$+\{a_1\sinh m(y+h)\sin m(ct-x)+2a_2\sinh 2m(y+h)\sin 2m(ct-x)$$
$$+3a_3\sinh 3m(y+h)\sin 3m(ct-x)\}^2].$$

Expanding the right hand side of this equation in powers of y and substituting the value of y as given by equation (5) we get

$$\left(\frac{Tm^2}{\rho c}+\frac{g}{c}\right)a_2\sinh mh\cos m(ct-x)+\left(\frac{4Tm^2}{\rho c}+\frac{g}{c}\right)a_2\sinh 2mh\cos 2m(ct-x)$$
$$+\left(\frac{9Tm^2}{\rho c}+\frac{g}{c}\right)a_3\sinh 3mh\cos 3m(ct-x).$$

$$=mc\Bigg[a_1\cos m(ct-x)\left\{1+\frac{m^2}{2!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^2\right.$$
$$\left.+\frac{m^4}{4!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^4+\ldots\ldots\right\}$$
$$+2a_2\cos 2m(ct-x)\left\{1+\frac{(2m)^2}{2!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^2\right.$$
$$\left.+\frac{(2m)^4}{4!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^{-4}+\quad+\ldots\ldots\right\}$$
$$+3a_3\cos 3m(ct-x)\left\{1+\frac{(3m)^2}{2!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^2\right.$$
$$\left.+\frac{(3m)^4}{4!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^4+\ldots\ldots\ldots\ldots\right\}\Bigg]$$

$$-\frac{m^2}{2}\Bigg[a_1\cos m(ct-x)\left\{1+\frac{m^2}{2!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^2\right.$$
$$\left.+\frac{m^4}{4!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^4+\ldots\ldots\ldots\ldots\right\}$$
$$+2a_2\cos 2m(ct-x)\left\{1+\frac{(2m)^2}{2!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^2\right.$$
$$\left.+\frac{(2m)^4}{2!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^4+\ldots\ldots\ldots\ldots\right\}$$
$$+3a_3\cos 3m(ct-x)\left\{1+\frac{(3m)^2}{2!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^2\right.$$
$$\left.+\frac{(3m)^4}{4!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh smh\cos sm(ct-x)\right)^4+\ldots\ldots\right\}\Bigg]^2$$

$$
\begin{aligned}
&-\frac{m^2}{2}\Bigg[a_1 \sin m\,(ct-x)\left\{m\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh\, smh\cos sm\,(ct-x)\right)\right.\\
&\qquad\left.+\frac{m^3}{3!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh\, smh\cos sm\,(ct-x)\right)^3+\ldots\ldots\right\}\\
&+2a_2 \sin 2m\,(ct-x)\left\{2m\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh\, smh\cos sm\,(ct-x)\right)\right.\\
&\qquad\left.+\frac{(2m)^3}{3!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh\, smh\cos sm\,(ct-x)\right)^3+\ldots\ldots\right\}\\
&+3a_3 \sin 3m\,(ct-x)\left\{3m\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh\, smh\cos sm\,(ct-x)\right)\right.\\
&\qquad\left.+\frac{(3m)^3}{3!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\sinh\, smh\cos sm\,(ct-x)\right)^3+\ldots\right\}\Bigg]
\end{aligned}
$$

Equating the coefficients of $\cos m\,(ct-x)$, $\cos 2m\,(ct-x)$ and $\cos 3m\,(ct-x)$ to zero, in the above equation we get the following approximate equations for the determination of c, a_2, and a_3, a_1 itself being arbitrary.

$$
\begin{aligned}
&\left(\frac{Tm^2}{\rho c}+\frac{g}{c}\right)\sinh mh-mc\cosh mh-\frac{3}{8}\frac{m^3}{c}a_1^2\sinh^2 mh\cos mh\\
&\qquad-\frac{5}{2}m^2a_2\sinh mh\sinh 2mh+m^2a_2\cosh 3\,mh=0\ldots\ldots\ldots \quad (6)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{4Tm^2}{\rho c}+\frac{g}{c}\right)a_2\sinh 2mh-2mca_2\cosh 2mh-2\frac{m^3}{c}a_1^2a_2\sinh^2 mh\cosh 2mh\\
&-\frac{m^3}{2c}a_1^2a_2\sinh mh\sinh 2mh\cosh 2mh+\frac{3}{2}\frac{m^3}{c}a_1^2a_2\sinh mh\sinh 3mh\\
&\qquad-\frac{m^3}{2}a_1^2\sinh^2 mh+\frac{m^2}{4}-\frac{m^3}{c}a_2^3\sinh^2 mh=0\ldots\ldots\ldots\ldots\ldots \quad (7)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{9Tm^2}{\rho c}+\frac{g}{c}\right)a_3\sinh 3mh-3mc\,a_3\cosh 3mh-\frac{5}{2}m^2a_1a_2\sinh mh\sinh 2mh\\
&\qquad-\frac{m^3}{2c}a_1^2a_3\sinh^2 mh\sinh 3mh-\frac{27}{4}\frac{m^3}{c}a_1^2a_3\sinh^2 mh\cosh 3mh\\
&+\frac{3m^3}{c}a_1^2a_3\sinh mh\sinh 4\,mh-\frac{4m^2}{c}a_2^2a_3\sinh 2mh\sinh 3mh\cosh 2mh\\
&\qquad-\frac{27}{4}\frac{m^3}{c}a_2^2a_3\sinh^2 2mh\cosh 3mh+\frac{15}{2}\frac{m^3}{c}a_2^2a_3\sinh 5mh\sinh 2mh\\
&\qquad-2a_2^2a_1\frac{m^3}{c}\cosh mh\sinh mh\sinh 2mh+\frac{3}{2}\frac{m^3}{c}a_1a_2^2\sinh mh\sinh 3mh\\
&+\frac{m^3}{c}a_1^2a_3\sinh 3mh\sinh 2mh+4\frac{m^3}{c}a_2^2a_3\sinh 4mh\sinh 3mh=0\ldots\ldots \quad (8)
\end{aligned}
$$

From equation (6) we get

$$c^2=\left(\frac{Tm}{\rho}+\frac{g}{m}\right)\tanh mh-\frac{3}{8}\frac{m^8}{c}a_1^2\ \sinh^2 mh-\frac{5}{2}m^2a_2\tanh mh\sinh 2mh$$
$$+m^2 a_2\frac{\cosh 3mh}{\cosh mh}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

From equation (7) we get

$$a_2=\frac{\frac{m^2}{2}a_1^2\sinh^2 mh-\frac{m^2}{4}}{\left(\frac{4Tm^2}{\rho c}+\frac{g}{c}\right)\sinh 2mh-2mc\cosh 2mh-\frac{2m^8}{c}a_1^2\sinh^2 mh\cosh 2mh-\frac{m^3}{2c}a_1^2\sinh mh\sinh 2mh\cosh mh+\frac{3}{2}\frac{m^8}{c}a_1^2\sinh mh\sinh 3mh.}$$
$$\ldots\ldots\ldots\ldots\ldots(10)$$

From equation (8) we have

$$a_3=\frac{\frac{5}{2}m^2a_1a_2\sinh mh\sinh 2mh+a_1a_2^2\frac{m^8}{c}\sinh^2 2mh-\frac{3}{2}\frac{m^8}{c}a_1a_2^2\sinh mh\sinh 3mh}{\left(\frac{9Tm^2}{\rho c}+\frac{g}{c}\right)\sinh 3mh-3mc\cosh 3mh-\frac{m^8}{2c}a_1^2\sinh^3 mh\sinh 3mh-\frac{27}{4}\frac{m^8}{c}a_1^2\sinh^2 mh\cosh 3mh+\frac{3m^8}{c}a_1^2\sinh mh\sinh 4mh-\frac{4m^8}{c}a_2^2\sinh 2mh\sinh 3mh\cosh 2mh-\frac{27}{4}\frac{m^3}{c}a_2^2\sinh^2 2mh\cosh 3mh+\frac{15}{2}m^8a_2^2\sinh 5mh\sinh 2mh.}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Hence

$$y=\frac{a_1}{c}\sinh mh\cos m\,(ct-x)$$
$$+\frac{\left(\frac{m^2}{2}a_1^2\sinh^2 mh-\frac{m^2}{4}\right)\sinh 2mh\cos 2m\,(ct-x)}{\left(\frac{4Tm^2}{\rho}+g\right)\sin 2mh-2mc^2\cosh 2mh-m^8a_1^2\left(2\sinh^2 mh\cosh 2mh+\ldots+\frac{1}{4}\sinh^2 2mh-\frac{3}{2}\sinh mh\sinh 3mh\right)}$$

$$\frac{3}{2}m^2a_1a_2\sinh mh\sinh 2mh+a_1a_2^2\frac{m^3}{c}\sinh^2 2mh$$

$$-\frac{3}{2}\frac{m^3}{c}a_1a_2^2\sinh mh\sinh 3mh$$

$$+\overline{\left(\frac{9Tm^2}{\rho}+g\right)\sinh 3mh\,\frac{3}{8}mc^2\cosh 3mh-m^3a_1^2}$$

$$\left(\frac{1}{2}a_1^2\sinh^2 mh\sinh 3mh+\frac{27}{4}+\ldots\ldots\ldots\right)$$

$$-m^3a_2^2\,(\ldots\ldots\ldots\ldots\ldots\ldots).$$

Putting $\frac{a_1}{c}\sinh mh=a$, the above formula becomes

$$y=a\cos m(ct-x)+\frac{\left(\frac{c^2m^2}{2}-\frac{m^2c^2}{4\sinh^2 mh}\right)\sinh 2mh\cos 2m\,(ct-x)}{\left(\frac{4Tm^2}{\rho}+g\right)\sinh 2mh-2mc^2\cosh 2mh-m^3a^2\left(2\cosh 2mh+\cosh^2 mh-\frac{3}{2}\frac{\cosh 3mh}{\cosh mh}\right)}$$

$$+\frac{\left\{\frac{5}{2}cm^2aa_2\sinh 2mh+aa_2^2m^3\left(\frac{\sinh^2 2mh}{\sinh mh}-\frac{3}{2}\sinh 3mh\right)\right\}\times\sinh 3mh\cos 3m\,(ct-x)}{\left(\frac{9Tm^2}{\rho}+g\right)\sinh 3mh-3mc^2\cosh 3mh-c^2m^3a^2\left(\frac{1}{2}\sinh 3mh+\frac{27}{4}\cosh 3mh-\frac{3}{2}\frac{\sinh 4mh}{\sinh mh}\right)-m^3a_2^2\left(2\sinh 4mh\sinh 3mh+\frac{27}{4}\sinh^2 mh\cosh 3mh-\frac{15}{2}\sinh 5mh\sinh 2mh\right.}$$

$$\ldots\ldots\ldots\ldots(12)$$

For infinite depth this formula becomes

$$y=a\cos m\,(ct-x)+\frac{\frac{m^2}{2}a^2c^2\cos 2m\,(ct-x)}{\left(\frac{4Tm^2}{\rho}+g\right)-2mc^2+\frac{1}{2}m^3a^2c^2-m^3a'^2c^2}$$

$$+\frac{\frac{5}{2}c^2m^2aa'-\frac{1}{2}c^2m^3aa'^2}{\left(\frac{9Tm^2}{\rho}+g\right)-3mc^2+\frac{3}{4}m^3c^2a^2+\frac{49}{4}m^3a'^2c^2}$$

$$\ldots\ldots\ldots\ldots\ldots(12a)$$

where
$$a' = \frac{\frac{m^2}{2}a^2}{\left(\frac{4Tm^2}{\rho} + g\right) - 2mc^2 + \frac{1}{2}m^3a^2c^2},$$

and
$$c^2 = \frac{Tm}{\rho} + \frac{g}{m}.$$

From this formula, the curves given below have been constructed to show the change of form as we increase the wave length.

In Fig. 7, $a = \cdot 2$ and $m = 3$

$y = \cdot 2 \cos m\,(ct - x) + \cdot 16 \cos 2m\,(ct - x) + \cdot 03 \cos 3m\,(ct - x)$

In Fig. 8, $a = \cdot 2$ and $m = 2 \cdot 5$

$y = \cdot 2 \cos m\,(ct - x) - \cdot 12 \cos 2m\,(ct - x) - \cdot 001 \cos 3m\,(ct - x)$

In Fig. 9, $a = \cdot 2$ and $m = 2 \cdot 25$

$y = \cdot 2 \cos m\,(ct - x) - \cdot 02 \cos 2m\,(ct - x) - \cdot 07 \cos 3m\,(ct - x)$

In Fig. 10, $a = \cdot 2$ and $m = 2$.

$y = \cdot 2 \cos m\,(ct = x) - \cdot 1 \cos 2m\,(ct - x) - \cdot 05 \cos 3m\,(ct - x)$.

FIG. 7.

FIG. 8.

FIG. 9.

FIG. 10.

The figures given in figures 7 to 10 correspond closely with those obtained experimentally for the same amplitudes and wave lengths. For practical purposes a depth of 4 or 5 cms. serves our purpose. The formula given above shows how these changes of

form come only between certain wave lengths, and how far large gravity waves, a_2 and a_3 become small and then the form becomes cycloidal,[1] as obtained by Stokes. The formula given above gives the form of small ripples as well as large gravity waves.

The formula (12) shows also the change of form of the waves with depth. From the numerator of the coefficient of cos $2m(ct-x)$ we can see how the change is more prominent for small values of m, i.e. for large values of λ.

We shall next discuss the case of the diverging waves mathematically.

SECTION V.—MATHEMATICAL THEORY FOR DIVERGING WAVES.

In this case we have to take cylindrical co-ordinates and take the axis of z vertically upwards and as before the undisturbed surface of the water as the plane, $z=0$.

The equations are modified as

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0 \quad \text{(13)}$$

$$\frac{\partial\phi}{\partial r}=0 \text{ when } r=\infty \quad \text{(14)}$$

$$\frac{\partial\phi}{\partial z}=0 \text{ when } z=-h \quad \text{(15)}$$

The pressure equation becomes

$$-\frac{T}{\rho}\frac{\partial^2 z}{\partial r^2}=-gz+\frac{\partial\phi}{\partial t}-\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial z}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2\right\} \quad \text{(16)}$$

We can therefore assume

$$\phi=\sum_{s=1}^{s=n} a_s J_0(s\,kr)\cosh sk(z+h)\sin s\,kct \text{ where } k=\frac{2\pi}{\lambda}. \quad \text{(17)}$$

and from the condition $-\frac{\partial\phi}{\partial z}=\frac{\partial z}{\partial t}$ we get for $z=0$

$$z=\frac{1}{c}\sum_{s=1}^{s=n} a_s J_0(s\,kr)\sinh s\,kh\cos s\,kct \quad \text{(18)}$$

Substituting these values in the pressure equation (16), and taking only three terms into consideration as in the case of plane waves we get

---

[1] For large gravity waves the formula reduces to

$$y=a\cos m(ct-x)-\frac{1}{2}ma^2\cos 2m(ct-x)+\frac{5}{8}m^2a^3\cos 3m(ct-x).$$

$$-\frac{Tk^2}{\rho c}\left[\sum_{s=1}^{s=3} s^2 a_s\, J_0\,(s\,kr)\ \sinh\, skh\ \cos\ skct\right]$$

$$= -\frac{g}{c}\sum_{s=1}^{s=3} a_s\, J_0\,(s\,kr)\sinh\, skh\ \cos\, skct + kc\sum_{s=1}^{s=3} sa_s\, J'\,(kr)\cosh\, sk\,(z+h)$$

$$\sin\, s\, kct$$

$$-\frac{k^2}{2}\left\{\left[\sum_{s=1}^{s=3} sa_s\, J_0\,(kr)\sinh s\, k\,(z+h)\sin\, s\, kct\right]^2\right.$$

$$\left.+\left[\sum_{s=1}^{s=3} sa_s\, J_0'\,(kr)\ \cosh\ s\, k\,(z+h)\ \cos\, s\, kct\right]^2\right\}$$

Expanding the right hand side of this equation in series of z and substituting the value of z given by (18) we get

$$\sum_{s=1}^{s=3}\left\{-\frac{s^2Tk^2}{\rho c}J_0''\,(s\,kr)+\frac{g}{c}J_0\,(s\,kr)\right\}a_s\ \sinh\ s\,kh\ \cos\ s\,kct$$

$$=kc\ a_1\ J_0\,(kr)\ \cos\ kct\left[1+\frac{k^2}{2!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^2\right.$$

$$\left.+\frac{k^4}{4!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^4+\ldots\right]$$

$$+\,2\,kc\,J_0\,(2\,kr)\ \cos\ 2\,kct\left[1+\frac{(2k)^2}{2\,!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^2\right.$$

$$\left.+\frac{(2\,k)^4}{4\,!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^4+\ldots\right]$$

$$+\,3\,kc\ J_0\,(3\,kr)\ \cos\ 3\,kct\left[1+\frac{(3k)^2}{2\,!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^2\right.$$

$$\left.+\frac{(3\,k)}{4\,!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^4+\ldots\right]$$

$$-\frac{k^2}{2}\left[a_1\ J'_0\,(kr)\sin\ kct\ \left\{1+\frac{k^2}{2}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^2\right.\right.$$

$$\left.+\frac{k^4}{4!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^2+\ldots\right\}$$

$$+\,2a_2\ J'_0\,(2\,kr)\ \sin\ 2\,kct\left\{1+\frac{(2\,k)^2}{2\,!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right.\right.$$

$$\left.+\frac{(2\,k)^4}{4!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^4+\ldots\right\}$$

$$+\,3a_3\ J'_0\,(3\,kr)\ \sin\ 3\,kct\left\{1+\frac{(3\,k)^2}{2\,!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^2\right.$$

$$\left.\left.+\frac{(3\,k)^4}{4\,!}\left(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\,J_0\,(s\,kr)\ \sinh\ s\,kh\ \cos\ s\,kct\right)^4+\ldots\right\}\right]^2$$

$$-\frac{k^2}{2}\Big[a_1\, J_0(kr)\sin kct\Big\{k\Big(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\, J_0(s\,kr)\sinh s\,kh\cos s\,kct\Big)$$

$$+\frac{k^3}{3!}\Big(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\, J_0(s\,kr)\sinh s\,kh\cos s\,kct\Big)^3+\ldots\Big\}$$

$$+2a_2\, J_0(2\,kr)\sin 2\,kct\Big\{2\,k\Big(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\, J_0(s\,kr)\sinh s\,kh\cos s\,kct\Big)$$

$$+\frac{(2\,k)^3}{3!}\Big(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\, J_0(s\,kr)\sinh s\,kh\cos s\,kct\Big)^2+\ldots\Big\}$$

$$+3a_3\, J_0(3\,kr)\sin 3\,kct\Big\{3\,k\Big(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\, J_0(s\,kr)\sinh s\,kh\cos s\,kct\Big)$$

$$+\frac{(3\,k)^3}{3!}\Big(h+\sum_{s=1}^{s=3}\frac{a_s}{c}\, J_0(s\,kr)\sinh s\,kh\cos s\,kct\Big)^3+\ldots\Big\}\Big]^2$$

Equating the co-efficients of cos kct, cos 2 kct and cos 3 kct to zero, we get the following approximate equations for the determination of c, a_2 and a_3.

$$-\frac{Tk^2}{\rho c}\, J_0''(kr)\sinh kh+\frac{g}{c}\, J_0(kr)\cosh kh-kc\; J_0(kr)\cosh kh$$

$$-\frac{k^2}{2c}\, a_1^2\,\{J_0(kr)\}^2\sinh^2 kh-\frac{3}{2}\frac{k^2}{c}\, a_2\, J_0(kr)\, J_0(2\,kr)\sinh mh\sinh 2\,mh$$

$$+\frac{k^2}{2}\, a_2\, J_1(kr)\, J_1(2\,kr)\cosh mh\cosh 2\,mh=0 \;\ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

$$-\frac{4Tk^2}{\rho c}\, a_2\, J_0''(2\,kr)\sinh 2\,kh+\frac{g}{c}\, a_2\, J_0(2kr)\sinh 2kh$$

$$-2\,kc\, a_2\, J_0(2\,kr)\cosh 2\,kh-\frac{3}{4}\, k^2a_1^2\{J_0(kr)\}^2\sinh^2 kh$$

$$+\frac{k^2}{4}\, a_1^2\{J_1(kr)\}^2\cosh^2 kh=0 \;\ldots\ldots\ldots\ldots\ldots \quad (20)$$

$$-\frac{9Tk^2}{\rho c}\, a_3\; J_0''(3\,kr)\sinh 3\,kh+\frac{g}{c}\, a_3\, J_0(3\,kr)\sinh 3\,kh$$

$$-3\,kc\, a_3\, J_0(3\,kr)\cosh 3\,kh-\frac{3}{2}\, a_1\, a_2\, J_0(kr)\, J_0(2\,kr)\sinh kh\sinh 2\,kh$$

$$+\text{etc.}\ldots=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

From (19) we have

$$c^2=-\frac{Tk}{\rho}\cdot\frac{J_0''(kr)}{J_0(kr)}\tanh kh+\frac{g}{k}\tanh kh-\frac{k}{2}\, a_1^2\,\{J_0(kr)\}^2\sinh kh\tanh kh$$

$$+k^2a_1\frac{J_1(kr)}{J_0(kr)}\, J_1(2\,kr)\cosh kh\cosh 2\,kh\,\frac{3}{2}\frac{k^2}{c}\, a_2\, J_0(2\,kr)\tanh kh\sinh 2\,mh.$$

Since $J_0''(kr)=-J_0(kr)+\frac{1}{kr}\, J_1(kr)$

We have as a first approximation

$$c^2=\left(-\frac{Tk}{\rho}\frac{J_0''(kr)}{J_0(kr)}+\frac{g}{k}\right)\tanh kh$$

$$=\left(\frac{Tk}{\rho}-\frac{Tk}{\rho}\cdot\frac{1}{kr}\frac{J_1(kr)}{J_0(kr)}+\frac{g}{k}\right)\tanh kh.$$

From (20) we have

$$a_2=\frac{\frac{3}{4}k^2a_1^2\{J_0(kr)\}^2\sinh^2 kh+\frac{k^2a_1^2}{4}\{J_1(kr)\}^2\cosh^2 kh}{-\frac{4Tk^2}{\rho c}J_0''(2kr)\sinh 2kh+\frac{g}{c}J_0(2kr)\sinh 2kh-2kcJ_0(2kr)\cosh 2kh}.$$

From (21) we have

$$a_3=\frac{\frac{3}{2}a_1a_2J_0(kr)J_0(2kr)\sin kh\sinh 2kh-a_1a_2J_1(kr)(J_2 kr)\cosh kh\cosh 2kh}{-\frac{9Tk^2}{\rho c}J_0''(3kr)\sinh 3kh+\frac{g}{c}J_0(3kr)\sinh 3kh-3J_0(3kr)\cosh 3kh-\frac{3}{2}a_1a_2J_0(kr)J_0(2kr)\sinh kh\sinh 2kh+\ldots}$$

Hence

$$z=\frac{a_1}{c}\sinh kh\,J_0(kr)\cos kct$$

$$+\frac{1}{c}\frac{\left[\frac{3}{4}k^2a_1^2\{J_0(kr)\}^2\sinh^2 kh+\frac{k^2a_1^2}{4}\{J_1(kr)\}^2\cosh^2 kh\right]\sinh 2kh\,J_0(2kr)\cos 2kct}{-\frac{4Tk^2}{\rho c}J_0''(2kr)\sinh 2kh+\frac{g}{c}J_0(2kr)\sinh 2kh-2kcJ_0(2kr)\cosh 2kh}$$

$$+\frac{1}{c}\frac{\frac{3}{2}a_1a_2J_0(kr)J_0(2kr)\sinh kh\sinh 2kh-a_1a_2J_1(kr)J_1(2kr)\cosh kh\cosh 2kh}{-\frac{9Tk^2}{\rho c}J_0''(3kr)\sinh 3kh+\frac{g}{c}J_0(3kr)\sinh 3kh-3kcJ_0(3kr)\cosh 3kh-\frac{3}{2}a_1a_2(\ldots.)}\times\sinh 3kh\,J_0(3kr)\cos 3kct$$

Putting $\frac{a_1}{c}\sinh kh=a$, and simplifying we get

$$z=aJ_0(kr)\cos kct$$

$$+\frac{\left[\frac{3}{4}c^2k^2a^2\{J_0(kr)\}^2+\frac{k^2}{4}c^2a^2\{J_1(kr)\}^2\coth^2 kh\right]\sinh 2kh\cos 2kct}{-\frac{4Tk^2}{\rho}\frac{J_0''(2kr)}{J_0(2kr)}\sinh 2kh+g\sinh 2kh-2kc^2\cosh 2kh}$$

$$+\frac{\frac{3}{2}ca\,a_2\,J_0(kr)\,J_0(2\,kr)\sinh 2\,kh - ca\,a_2\,J_1(kr)\,J_1(2\,kr)\coth kh\cosh 2\,kh \times \sin 3\,kh\,J_0(3\,kr)\cos 3\,kct}{-\frac{9Tk^2}{\rho}J_0''(3\,kr)\sinh 3\,kh + g\,J_0(3\,kr)\sinh 3\,kh - 3\,c^2k\,J_0(3\,kr)\cosh 3\,kh}$$

For infinite depth this formula reduces to

$$z = a\,J_0(kr)\cos kct + \frac{\left[\frac{3}{4}c^2k^2a^2\{J_0(kr)\}^2 + \frac{c^2k^2a^2}{4}\{J_1(kr)\}^2\right]\cos 2\,kct}{\left(-\frac{4Tk^2}{\rho}\frac{J_0''(2\,kr)}{J_0(2\,kr)} + \frac{g}{c}\right) - 2\,kc^2 + \ldots.}$$

$$+\frac{\frac{3}{2}ca\,a_2\,J_0(kr)\,J_0(2\,kr) - ca\,a_2\,J_1(kr)\,J_1(2\,kr)}{-\frac{9Tk^2}{\rho}\frac{J_0''(3\,kr)}{J_0(3\,kr)} + g - 3\,kc^2 - \frac{3}{2}a\,(\ldots\ldots\ldots)}\cos 3\,kct.$$

The asymptotic values of $J_0(kr)$ is $\sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{\pi}{4}\right)$. and that of $J_1(kr)$ is $\sqrt{\frac{2}{\pi kr}}\sin\left(kr - \frac{\pi}{4}\right)$

Substituting these values in the above formula, we get

$$z = a\sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{\pi}{4}\right)\cos kct + \frac{2}{\pi kr}\cdot\frac{\frac{1}{4}k^2a^2(1 + 2\sin 2\,kr)}{-\frac{4Tk^2}{\rho}\frac{J_0''(2\,kr)}{J_0(2\,kr)} + g - 2\,kc^2}\cos 2\,kct$$
$$+ \text{etc.}$$

The above formula shows that the amplitude of the second harmonic decreases inversely as the distance from the origin, whereas the fundamental decreases inversely as the square root of the distance. Hence the wave form would tend to become simpler with the increase of distance from the source. This is what is also found experimentally.

SECTION VI.—DEPENDENCE OF THE VELOCITY OF RIPPLES ON AMPLITUDE AND DEPTH OF LIQUID.

$$c^2 = \left(\frac{Tm}{\rho} + \frac{g}{m}\right)(1 + \alpha m^2a^2).$$

In the case of waves of finite amplitude, α becomes a positive fraction. The value of α can also be easily deduced[1] for large waves

[1] Lamb's Hydrodynamics (4th edition), p. 410.

by taking only one term in the series and by using the method of successive approximation.

In order to find the velocity of propagation experimentally, it is necessary to find the absolute frequency of the motor. This can be easily measured by means of an electric tachometer. But for purposes of comparison it is not necessary to find the absolute frequency. Since the velocity is proportional to wave-length, if the frequency remains constant, it will be sufficient to measure only the wave-lengths. The amplitudes of oscillation are measured by means of a travelling microscope or simply by means of the scale in the eye piece of a low power microscope. The wave lengths are measured on the screen and knowing the ratio of the length of a line on the surface of water and its image on the screen we can find the absolute value of the wave length of the ripples. In taking these measurements, it is absolutely necessary to keep the frequency of the motor constant, or otherwise the results are not comparable. This can be easily secured by means of the rheostat which is included in the circuit of the motor. The frequency can be read off directly from the electric tachometer. Some observations made in this manner show that the velocity of long waves increases with the amplitude as indicated by the theory. A more detailed comparison between theory and experiment is at present being carried out.

Neglecting for the time being the effect of amplitude we get

$$c^2 = \left(\frac{Tm}{\rho} + \frac{g}{m}\right) \tanh mh$$

This form is well known and it is sufficient here simply to indicate that this value of c^2 agrees with the experimental results when the amplitude is very small. From the formula it can be seen that for small wave lengths the velocity continues to be constant till a very low depth of a few millimeters is reached. But for large wave lengths, i.e. for small vales of m, the increase in the value of c^2 becomes apparent even for greater depths. This result is in agreement with what is actually observed.

SECTION VII.—OUTLINE OF FURTHER RESEARCH TO BE CARRIED OUT.

The preceding section describes the work that has been carried out so far. There is much that still remains to be done. In the

present investigation regarding the form of ripples, we have not taken into account the effect of viscosity. The chief effect of viscosity is to diminish the amplitude and this in its turn affects the form of the wave. It appears also likely that the variation of the velocity with amplitude may have a marked effect on the group velocity of ripples of finite amplitude. It will be interesting to investigate this experimentally. In the theory of group velocity usually given, the effect of finite amplitude on the speed of the waves is not taken into account, and it seems possible that this may considerably influence the results. As we have seen, the effect of increasing the amplitude beyond a certain point is to cause the waves to divide and it would be interesting to make direct measurements of the amplitude of the waves in such cases and to compare the same with theory. The dependence of ripple form on depth, especially in those cases in which the tank is of variable depth also appears to merit fuller investigation. The vessel in which the liquid is contained might, for instance, be wedge-shaped or hemispherical, and so on. Experiments with other liquids might also be carried out.

SECTION VIII.—SYNOPSIS.

It was shown in the previous sections how the form of ripples of different wave lengths changes with the amplitude and frequency. To study these forms Fleming's motor vibrator as improved by Prof. Raman was used to excite the ripples. This vibrator has many advantages over the ordinary electrically maintained tuning fork, viz. the amplitude of vibration and the frequency can be regulated to a nicety. The bottom of the tank is made of a glass plate and is illuminated by intermittent light, so that the ripples appear stationary.

It was found that when the amplitude is small, the ripples were simple undulations for all wave lengths, but for finite amplitudes, with the increase of wave length, the crests of the waves become double, triple, and so on. With increasing distance from the source, these forms become simpler, as also with decreasing depth of the liquid in the tank. A mathematical theory explaining all these phenomena is given, both for plane and diverging waves.

It is also shown that the velocity of large waves increases with amplitude and the mathematical results are at present being

verified. The way in which the velocity depends upon the depth is also worked out, and there it is shown that the velocity decreases with the depth, though the change is not perceptible in the case of smaller waves till a very smaller depth is reached.

In conclusion, the author wishes to express his best thanks to Prof. C. V. Raman who suggested the investigation for the facilities afforded and for his constant encouragement and interest during the progress of the research.

XV. The Effects of a Magnetic Field on the General Spectrum.

By H. P. Waran, M.A., Government Scholar of the University of Madras.

(Plate XIII, Fig. 2.)

The complex change taking place in the source when the radiation is emitted in a magnetic field does not seem to have had the attention it deserves. Though the spectrum shows a variety of changes under the influence of the field, yet the study of the spectrum has mainly been confined to the Zeeman effect on a particular line isolated from the rest of the spectrum.

Some remarkable changes observed in the case of a mercury discharge-tube brought the phenomenon to my notice and some aspects of the problem with the rare gases were experimentally worked out and some of the very interesting results obtained summarised in a paper [1] read before the Cambridge Philosophical Society.

Since then my attention has been drawn to the fact that some work has already been done on the subject but without coming to any definite conclusions. As early as 1870 Rand Capron in his book on Aurora mentions some experiments with a discharge-tube placed in a magnetic field, and speaks of some lines being brought out by it.

The latest paper [2] on the question has been by Norton A. Kent and Royal M. Frye, where a bibliography of the past work on the subject is also given. In this paper though the authors set out with the idea of elucidating the real nature of the phenomenon,

[1] Proceedings, Cambridge Philosophical Society, 1920.
[2] *Astrophysical Journal*, Vol. 37 (1913), p. 183.

the experimental conditions they adopted seem to have led them to doubtful conclusions. In the main they attribute the changes observed to the disruptive action of the discharge when it is deflected by the field into the walls of the tube, the changes in the spectrum being due to the products of dissociation of the glass. Though this conclusion is fairly justifiable from their experiments, yet there are a few aspects of the problem which are not so easily explained. In fact my observations point to the contrary effect,

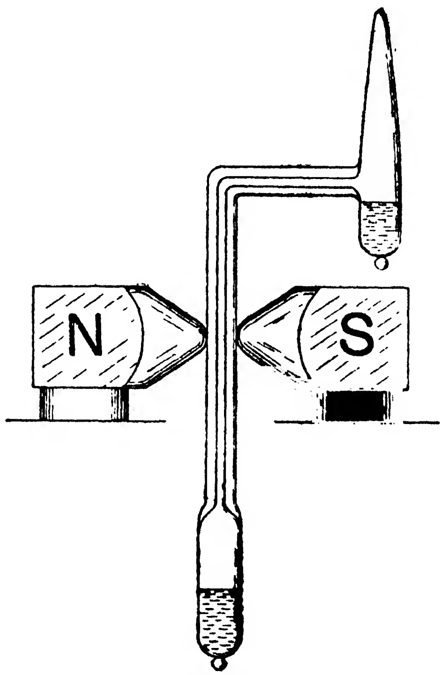
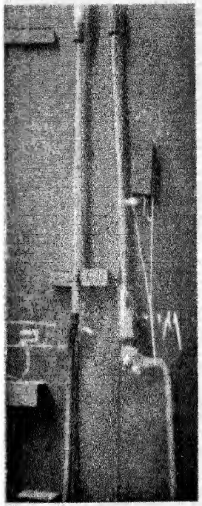


FIG. 1.

that this dissociation of the glass is purely a side issue and a natural consequence when powerful fields and discharges are used. In practice by the employment of such powerful fields and discharges the real effect gets masked and complicated to a considerable degree. To examine this effect of the magnetic field on the radiation, the disruptive action should be reduced to a minimum by the employment of a moderate field and current and even then the tube must be of a material like quartz not liable to suffer such easy decomposition arising from any slight local heating.



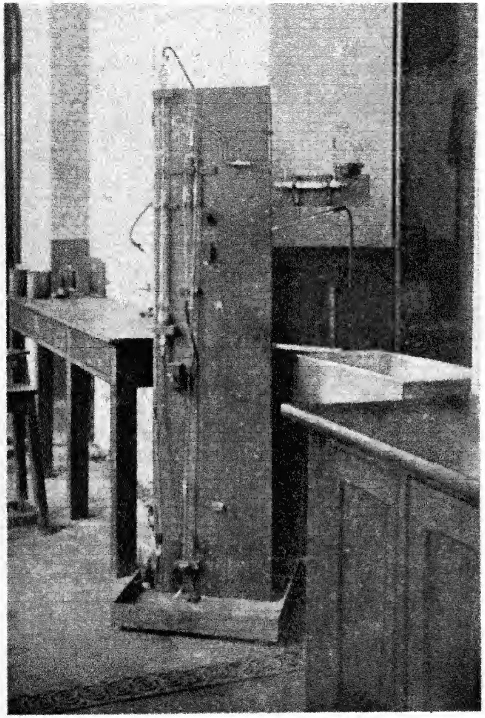
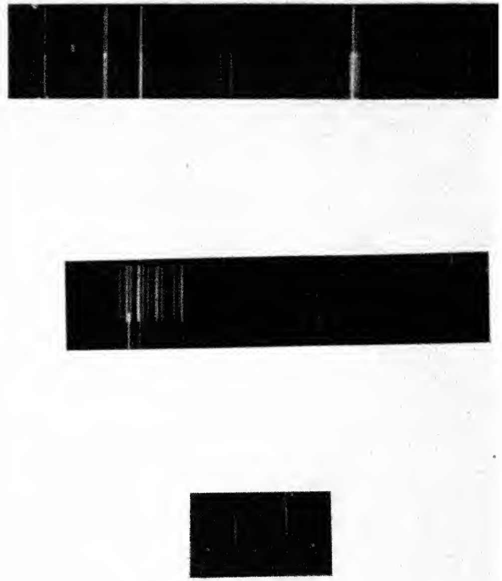


Fig. 1 (a) Fig. (I)

An Automatic Mercury Pump



(a) off on Hg

(b) on off Ne & H

(c) off on He, O & H

Fig. 2
Effect of Magnetic Field on General Spectrum,

In my experiments the diameter of the quartz capillary was about 1·5 mm. and the current about 2 milliamperes, the field being of the order of about 5000 C.G.S. units. The arrangement adopted is illustrated in Fig. 1, and the results obtained point to a definite and positive effect of the magnetic field whose nature, however, is not yet quite clear.

In the case of a mercury discharge-tube at a rather low pressure the four lines in the red

$$\lambda\lambda\ 6234,\ 6152,\ 6123,\ 6072$$

appear faintly. But the effect of the magnetic field on these four is different. The line λ 6152 is enhanced very much while the others decrease in brilliancy. In fact at a certain stage the line λ6152 is practically invisible and is then brought out very brilliantly by the magnetic field. Such an anomalous behaviour in a set of lines belonging to the same element can be attributed only to a positive effect of the magnetic field. It may also be mentioned here that this line λ6152 of mercury has already appeared abnormal, in that it is a line brought out when helium is present in the tube. This line seems to have some peculiar significance in the spectrum of mercury and deserves a special study. Fig. 2 (*a*) in the Plate XIII shows the effect in the case of the mercury spectrum and this abnormal line is marked by a dot.

A line of argument to explain this effect led to experiments with the rare gases helium, neon, etc., and their mixtures with the diatomic gases oxygen, hydrogen, etc. The effect of the magnetic field was to enhance very considerably the lines of these monatomic gases, even when they are present in a pure state. When mixed with other diatomic gases the remarkable point is that the lines of those monatomic gases alone are considerably enhanced, while those of the diatomic and others are scarcely affected. Thus in a tube of neon and hydrogen the principal hydrogen lines are not affected at all, while the neon lines are very much enhanced [Fig. 2 (*b*) in the Plate]. Similarly when a trace of helium is mixed with hydrogen or oxygen and the helium lines are ordinarily not visible at all, the magnetic field brings out the helium lines prominently, leaving the others scarcely affected [Fig. 2 (*c*)]. These results led to the natural inference by analogy that such enhanced lines are the radiations of an atom while the others may be attributed to the molecule.

The reason why the atomic radiation should be enhanced is not yet known. Thus in the case of mercury the enhanced lines $\lambda 6152$ and others are evidently to be classified as atomic radiations and the unaffected lines are due to the molecules. In such a view it is not meant that these are absolutely pure radiations. It may be that every line is made up of the radiations of both the atom and the molecule, but that the proportions are different in different cases, so that we may explain the highly enhanced radiations as those in which the atomic radiation predominates. The enhancement of the argon lines noted by Kent and Frye is also in keeping with this explanation.

This line of argument led to experiments with sulphur and iodine in a vacuum tube and as is to be expected from the complex of molecules and atoms in all states of aggregation which we know to exist there, the spectrum, under the influence of the magnetic field, showed a variety of marked changes. These cases are under more detailed study at present. Even in the case of the diatomic gases the effects are not simple, but considerably complicated, since their atomic or molecular state and the proportionality of their radiations can be expected to be dependent on pressure and temperature. Even in the case of atmospheric air at low pressure some marked changes are observed identical with quartz or glass tubes, so that the question of the disintegration products of the glass and its complications do not arise at all. Further examination of the phenomenon in greater detail is in progress.

Cavendish Laboratory,
Cambridge, England;
May 6th, 1920.

XVI. An Improved Type of Automatic Mercury Pump.

By H. P. Waran, M.A., Government Scholar of the University of Madras.

(Plate XIII, Fig. 1 and 1(*a*).)

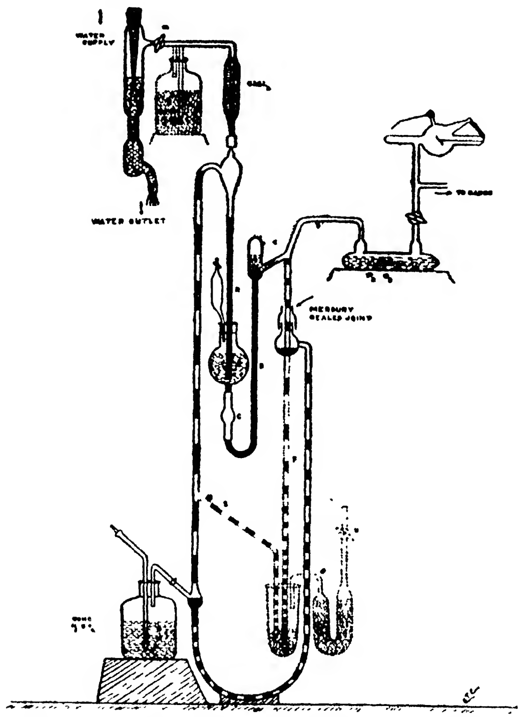
The pump under consideration is of a German design and worked by the mercury falling in drops down a long fall tube catching small pellets of air between as per original sprengel principle. The pump is automatic in action since the mercury that fell down to the bottom reservoir is transferred back again to the upper reservoir and made to flow again with the help of a filter pump worked off the main water-supply.

A quick acting pump of the type was urgently required in connection with some cathodic deposition experiments and when the pump was constructed as per original design it was found not to work at all. Further experimentation with it showed the necessity for a good deal of improvement to be made on the original design before it could be made working efficiently.

Described in brief the original design of the pump was as follows:—

A wide-necked flask-like vessel *A* with a tubulure at the bottom is the main mercury reservoir and it contains about 2lb. of mercury. On the rubber connection from the tubulure downwards to the bend of the air trap tube *B* is placed a clamp *C* to regulate the flow of mercury. The mouth of *A* is closed with a rubber stopper through which passes the central mercury return tube *R* and a drying tube of calcium chloride. From the bottom of the air trap *A* leads up a tube which branches upwards to the receiver connections and downwards as the main fall tube *F*. This fall tube is about 4 to 4½ ft. in length and terminates in a lower

reservoir, a big stout test tube closed by a three-hole rubber cork through which a side suction tube S as well as an air inlet drying tube D also passes. The suction tube goes up along the side and connects to the top of the mercury return tube R. From the top bulb of this return tube R, an exhaust tube with cock X is led through a drying bottle P of concentrated sulphuric acid, to the suction pipe of a filter pump. The tube Y to the receiver goes along



[The dotted lines indicate portions of the original design which do not appear in the finally modified form of the instrument.]

a phosphorus pentoxide tube and terminates at the receiver (a discharge tube) with a side connection to the vacuum gauge.

Now when the cock IV is closed, the filter pump started and the cock X opened, air is rapidly sucked away fom the whole system until it reaches the limit of suction of the pump, i.e. about 2 centimetres of mercury pressure. Then the clamp C is gradually opened and mercury run into from the trap at T and fill up the

bottom test tube reservoir up to the mouth of S. Then the rate of fall of mercury is reduced with C and cock W is opened when a vigorous suction of air is maintained up S and all excess mercury coming down F in pellets is sucked up along with alternate pellets of air into the tap of Q wherefrom it reaches the main reservoir A vertically below. Thus the mercury is maintained in automatic circulation and the pellets of mercury going down the fall tube do the exhaustion. This was the original design of the pump.

When such a pump was made and tried it was not found to work quite efficiently and at first the mercury could not be got into automatic circulation at all. Immediately experiments to set it going were begun after temporarily disconnecting the air trap and other accessories.

Instead of a single filter pump a pair of filter pumps were connected in parallel, and even though the upward draught was much more powerful the mercury could not be got into automatic circulation. Further thought on the subject suggested that the main essential point was the necessity to establish a very strong air suction through the side tube to enable the mercury to be sucked up as rapidly as it comes down the central main fall tube. To secure this the vacuum created by the filter pump should be high and speedy and the main drawback of the original design was detected to be the large capacity introduced in the air suction circuit. The drying bottle of concentrated sulphuric acid as well as the bottom reservoir were too big, and once the bottom of the tube S got uncovered and the full atmospheric pressure of air rushed in, it took some time for the pump to establish the original vacuum and a powerful draught to take up the mercury from the lower reservoir. It naturally follows that the smaller this capacity, the more quickly the vacuum created and the more powerful the draught. So a small drying tube of calcium chloride is put in between the pump and the suction tube, while the main drying bottle of concentrated sulphuric acid is put in lower down to enable the air to get dried before being sucked into the system. The next improvement was the abolition of the lower reservoir altogether wich was an improvement in not only that it decreased the lower capacity still further, but eliminated the difficulty of a 3-holed rubber cork with air leaks and so on at this reservoir. Instead of this reservoir the lower end of the fall tube was curved and made to join up the suction tube

with a small bulb at the beginning of the suction tube with a tap sealed on to it. This tap leads to the drying sulphuric acid bottle and the suction tube of this bottle has another tap sealed on to it partly for regulation and partly to close up the acid from external moisture when the pump is left idle. This improvement prevented the accumulation of mercury at the bottom owing to any maladjustment of the rate of fall and suction, since the moment the mercury level rose up in the bulb to close the air inlet, the excess was sucked up in the powerful draught and then returned to the main reservoir through the tube *R*. With this modification the automatic mercury circulation worked up quite quickly and very vigorously even when only one ordinary locally made glass filter pump was used. The air getting mixed up with the mercury is quite dry and no fouling of the vacuum from vapour pressure is to be expected. It is very useful to observe the general precaution that the bore of the suction tube *S* as well as that of the fall tube should be about a millimetre and not more, to prevent the air escaping by the side of the mercury pellets. These tubes are conveniently made of thick walled capillary glass tubing of internal diameter about a millimetre.

The vacuum air trap is as usual and is an essential adjunct when the highest vacuum is to be used. With these modifications introduced the pump works quite easily and rapidly, perfectly automatic in action, once the rate of fall of mercury is adjusted and the two taps at the bottom opened after preliminary vacuum of about 2 centimetres of mercury has been directly obtained by the filter pump. Quite a cathode ray vacuum is produced in about 30 or 40 minutes everything going well. But working the pump longer, the vacuum was found not to improve further, and it was difficult to push the discharge tube to quite a hard condition. An examination in detail of what is happening in the fall tube at this stage shows the reason why. The fall tube must be scrupulously clean and when the metallic clicks of the falling mercury pellets are noticed it will be found that a good length of air in the fall tube is compressed by the falling column of mercury to quite a tiny disc of air of imperceptible size. This air somehow does not seem to escape out of the tube along with the mercury, but somehow escapes back into the high vacuum system, probably escaping gradually by the side of the falling pellets. This difficulty can be

got over if we can prevent the air being compressed into this thin disc-like form.

A suggestion to get over this difficulty is to do this latter high exhaustion in two stages so that in the first stage the air is not compressed but simply pushed into another highly exhausted space to produce the extreme vacuum. Following this idea the fall was made in two stages, the first of about 30 centimetres and the other of about 90 centimetres, and the tip (lower) of the first tube was just touching the mercury level on the reservoir on top of the lower tube as illustrated. In such a system, at the initial stages when the pressure is high, the first fall is not very effective and the real exhaustion takes place by the lower fall tube as usual. But later as the vacuum gets very high and the mercury begins to hammer in the lower fall tube, the rate of flow of mercury is slightly increased to secure two or three pellets going down the first tube at the same instant. This succession of pellets down the first fall are very effective and they push the remaining air from the receiver into the upper reservoir from which they can never escape into the high vacuum so long as the downward stream of mercury continues and is maintained permanently by the filter pump. The little air that collects in the upper reservoir is of course removed whenever it is an appreciable quantity by the lower fall tube. This is the most important of all the improvements and its effect is quite remarkable. With its help the pump exhausts to limits which it could not previously exhaust and in the case of one small locally made X-ray tube with electrodes about 2 inches apart, the discharge could be made to prefer an alternate air gap of about 4 or 5 inches with a six-inch coil.

The above piece of work was done at the Presidency College, Madras, and Fig. 1 in Plate XIII shows the pump during its experimental stages when just got working automatically. Fig. 1(*a*) is a photograph of the double stage exhaustion attachment for high vacuum work.

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